24. $\text{rref}\begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Here our kernel is the span of only one vector:

$\left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$, while a basis of the image of A is $\left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix}\right)$.

26. a. We notice that each of the six matrices has two identical columns. In matrices C and L, the second column is identical to the third, so that $\ker(C) = \ker(L) = \text{span}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.
In matrices H, T, X and Y, the first column is identical to the third, so that $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Thus, only L has the same kernel as C.

b. We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix C are identical, so that any vector $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\text{im}(C)$ will satisfy the equation $y_1 = y_3$. We can conclude that $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3\right\}$, $\text{im}(L) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2\right\}$, and $\text{im}(T) = \text{im}(Y) = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3\right\}$.

c. Our discussion in part b shows that the answer is matrix L.

32. We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\left|\begin{array}{ccccccc} x_1 & & & - & x_3 & + & x_4 & = & 0 \\ & & x_2 & + & 2x_3 & + & 3x_4 & = & 0 \end{array}\right|$, which in turn amounts to finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using Kyle Numbers, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

38. a. The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\ker(A)) = 5 - \text{rank}(A)$ is 2,3,4, or 5.

b. The rank of a 7×4 matrix A is at most 4, so that $\dim(\text{im}(A)) = \text{rank}(A)$ is 0,1,2,3, or 4.

52. $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By Exercises 50 and 51a, $[1\ 0\ -1\ -2]$, $[0\ 1\ 2\ 3]$ is a basis of the row space of A.

36. No; if $\mathrm{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\mathrm{im}(A)) = 2\dim(\mathrm{im}(A))$, so that n is an even number.

56. Using the terminology suggested in the Exercise, we multiply the relation $c_0\vec{v} + c_1A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ with A^{m-1} and obtain $c_0A^{m-1}\vec{v} = \vec{0}$ (all other terms vanish since $A^m = 0$).

Since the vector $A^{m-1}\vec{v}$ is nonzero (by construction), the scalar c_0 must be zero, and our relation simplifies to $c_1A\vec{v} + c_2A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$.

Now we multiply both sides with A^{m-2} and obtain $c_1A^{m-1}\vec{v} = \vec{0}$, so that $c_1 = 0$ as above. Continuing like this we conclude that all the c_i must be zero, as claimed.