

6. a. Yes!

- The zero vector is in $V \cap W$, since $\vec{0}$ is in both V and W .
 - If \vec{x} and \vec{y} are in $V \cap W$, then both \vec{x} and \vec{y} are in V , so that $\vec{x} + \vec{y}$ is in V as well, since V is a subspace of \mathbb{R}^n . Likewise, $\vec{x} + \vec{y}$ is in W , so that $\vec{x} + \vec{y}$ is in $V \cap W$.
 - If \vec{x} is in $V \cap W$ and k is an arbitrary scalar, then $k\vec{x}$ is in both V and W , since they are subspaces of \mathbb{R}^n . Therefore, $k\vec{x}$ is in $V \cap W$.
- b. No; as a counterexample consider $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in \mathbb{R}^2 .

20. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is redundant, simply because it is the zero vector.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is our first non-zero vector, and thus, is not redundant.

$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and is redundant.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and is not redundant.

$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and is redundant.

Similarly, $\begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and is also redundant.

However, by inspection, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, meaning that this last vector is not redundant. Thus, the seven vectors are linearly dependent.

24. $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ is redundant, because $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Thus, $3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0}$

and $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ is in the kernel of $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$.

28. The three column vectors are linearly independent, since $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} = I_3$.

Therefore, the three columns form a basis of $\text{im}(A) (= \mathbb{R}^3)$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}.$$

Another sensible choice for a basis of $\text{im}(A)$ is $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

48. We can write $3x_1 + 4x_2 + 5x_3 = [3 \ 4 \ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, so that $V = \ker[3 \ 4 \ 5]$.

To express V as an image, choose a basis of V , for example, $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}$.

$$\text{Then, } V = \text{im} \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}.$$

There are other solutions.

36. Yes; we know that there is a nontrivial relation $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m = \vec{0}$.

Now apply the transformation T to the vectors on both sides, and use linearity:

$$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m) = T(\vec{0}), \text{ so that } c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_m T(\vec{v}_m) = \vec{0}.$$

This is a nontrivial relation among the vectors $T(\vec{v}_1), \dots, T(\vec{v}_m)$, so that these vectors are linearly dependent, as claimed.

38. a. Using the terminology introduced in the exercise, we need to show that any vector \vec{v} in V is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$. Choose a specific vector \vec{v} in V . Since we can find no more than m linearly independent vectors in V , the $m+1$ vectors $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$ will be linearly dependent. Since the vectors $\vec{v}_1, \dots, \vec{v}_m$ are independent, \vec{v} must be redundant, meaning that \vec{v} is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$, as claimed.
- b. With the terminology introduced in part a, we can let $V = \text{im}[\vec{v}_1 \ \cdots \ \vec{v}_m]$.