12. Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span}\left(\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\\0\end{bmatrix}\right)$.

22. Compare with the solution to Exercise 21.

$$\left[\begin{array}{cc:c}2 & 1 & 3\\3 & 4 & 2\\6 & 5 & 7\end{array}\right] \longrightarrow \left[\begin{array}{cc:c}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{array}\right]$$

This computation shows that the third column vector of A, $\vec{v}_3$, is a linear combination of the first two, Thus, only the first two vectors are independent, and the image is a plane in $\mathbb{R}^3$.

34. To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 \quad \text{and} \\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\2x+z\end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

44. a. Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b. No; as a counterexample, consider $A = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.

54. a. If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.

If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that

$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i$th column of H.

Since the columns of H are all different, this method allows us to find out where an error occurred.

b. $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

Therefore $\vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

38. a. If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$.

Therefore, $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \dots$

Exercise 37 shows that these kernels need not be equal.

b. If a vector $\vec{y}$ is in $\operatorname{im}(A^{k+1})$, that is, $\vec{y} = A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\operatorname{im}(A^k)$, since we can write $\vec{y} = A^k(A\vec{x})$. Therefore, $\operatorname{im}(A) \supseteq \operatorname{im}(A^2) \supseteq \operatorname{im}(A^3) \supseteq \dots$.

Exercise 37 shows that these images need not be equal.

48. a. $\vec{w} = A\vec{x}$, for some $\vec{x}$, so that $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$.

b. If $\operatorname{rank}(A) = 2$, then A is invertible, and the equation $A^2 = A$ implies that $A = I_2$ (multiply by A^{-1}).

If $\operatorname{rank}(A) = 0$ then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

c. First note that $\operatorname{im}(A)$ and $\ker(A)$ are lines (there is one nonleading variable).

By definition of a projection, we need to verify that $\vec{x} - A\vec{x}$ is in $\ker(A)$. This is indeed the case, since

$A(\vec{x} - A\vec{x}) = A\vec{x} - A^2\vec{x} = A\vec{x} - A\vec{x} = \vec{0}$ (we are told that $A^2 = A$). See Figure 3.5.

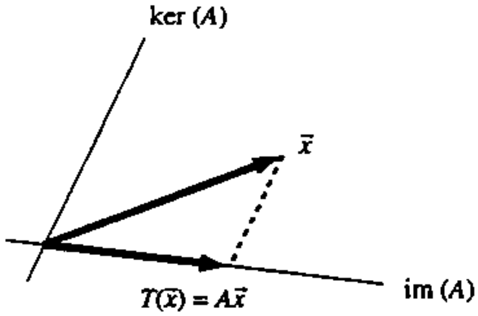

Figure 3.5: for Problem 3.1.48c.

46. If $\text{rank}(A) = r$, then $\text{im}(A) = \text{span}(\vec{e}_1, \ldots, \vec{e}_r)$. See Figure 3.4.

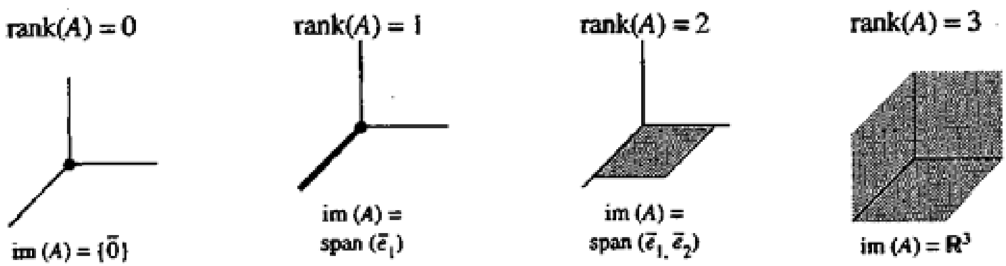

Figure 3.4: for Problem 3.1.46.