6. By Fact 2.2.1, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

$\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}$.

5. From Definition 2.2.2, we can see that this is a reflection about the line $x_1 = -x_2$.

16. a. See Figure 2.20.

b. By Fact 2.1.2, the matrix of T is $[T(\vec{e}_1) \quad T(\vec{e}_2)]$.

$T(\vec{e}_2)$ is the unit vector in the fourth quadrant perpendicular to $T(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$, so that

$T(\vec{e}_2) = \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}$. The matrix of T is therefore $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Alternatively, we can use the result of Exercise 13, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ to find the matrix

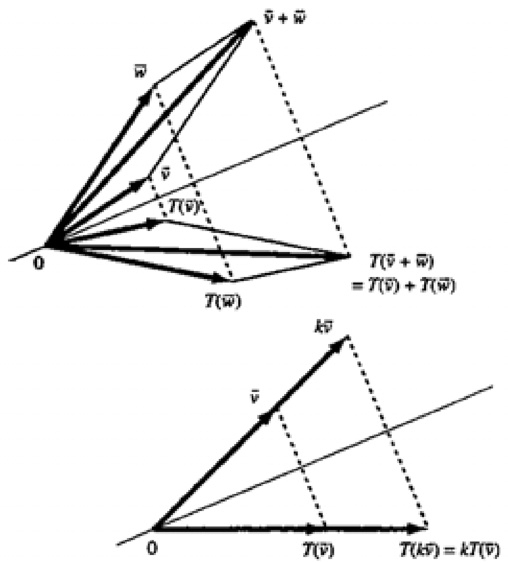

Figure 2.20: for Problem 2.2.16a.

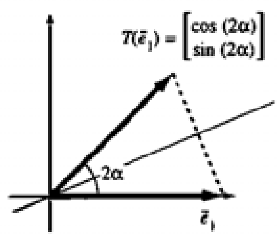

Figure 2.21: for Problem 2.2.16b.

$$\begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}.$$

You can use trigonometric identities to show that the two results agree.

30. Write $A = [\vec{v}_1 \quad \vec{v}_2]$; then $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$. We must choose $\vec{v}_1$ and $\vec{v}_2$ in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2$ is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all x_1 and x_2. This is the case if (and only if) both $\vec{v}_1$ and $\vec{v}_2$ are scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For example, choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

34. Keep in mind that the columns of the matrix of a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

If T is the orthogonal projection onto a line L, then $T(\vec{x})$ will be on L for all $\vec{x}$ in $\mathbb{R}^3$; in particular, the three columns of the matrix of T will be on L, and therefore pairwise parallel. This is the case only for matrix B: B represents an orthogonal projection onto a line.

A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E: E represents the reflection about a line.

47. Write $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$.

a. $f(t) = \left(T\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix} \cdot \begin{bmatrix} -a\sin t + b\cos t \\ -c\sin t + d\cos t \end{bmatrix}$

$= (a\cos t + b\sin t)(-a\sin t + b\cos t) + (c\cos t + d\sin t)(-c\sin t + d\cos t)$

This function $f(t)$ is continuous, since $\cos(t)$, $\sin(t)$, and constant functions are continuous, and sums and products of continuous functions are continuous.

b. $f\left(\frac{\pi}{2}\right) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\left(T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, since T is linear.

$f(0) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The claim follows.

c. By part (b), the numbers $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have different signs (one is positive and the other negative), or they are both zero. Since $f(t)$ is continuous, by part (a), we can apply the intermediate value theorem. (See Figure 2.33.)

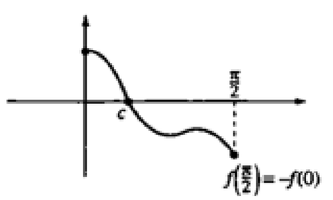

Figure 2.33: for Problem 2.2.47c.

d. Note that $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ are perpendicular unit vectors, for any t. If we set

$\vec{v}_1 = \begin{bmatrix} \cos(c) \\ \sin(c) \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -\sin(c) \\ \cos(c) \end{bmatrix}$, with the number c we found in part (c), then $f(c) = T(\vec{v}_1) \cdot T(\vec{v}_2) = 0$, so that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular, as claimed. Note that $T(\vec{v}_1)$ or $T(\vec{v}_2)$ may be zero.

50. Use the hint: Since the vectors on the unit circle are of the form $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$, the image of the unit circle consists of the vectors of the form $T(\vec{v}) = T(\cos(t)\vec{v}_1 + \sin(t)\vec{v}_2) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$.

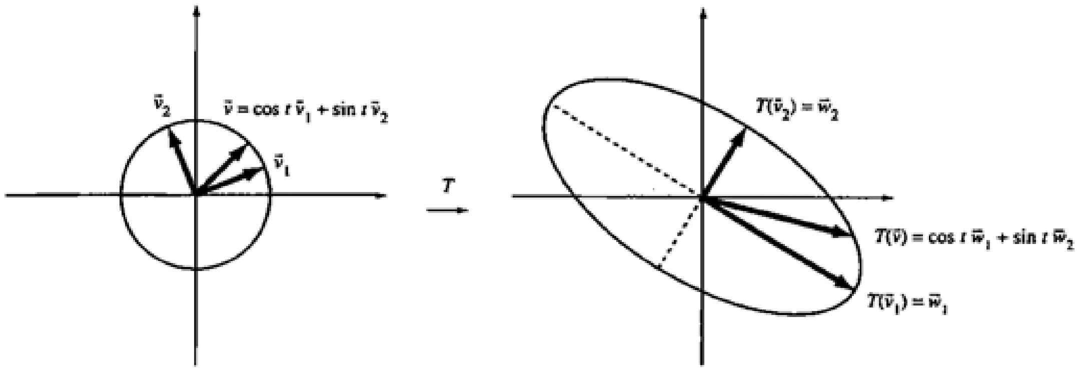

Figure 2.37: for Problem 2.2.50.

These vectors form an ellipse: Consider the characterization of an ellipse given in the footnote, with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$. The key point is that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular.