6. Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

14. a. By Exercise 13a, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ is invertible if (and only if) $2k - 15 \neq 0$, or $k \neq 7.5$.

b. By Exercise 13b, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.

If all entries of this inverse are integers, then $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is a (nonzero) integer n, so that $2k - 15 = \frac{1}{n}$ or $k = 7.5 + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$ is an integer as well, n must be odd.

We have shown: If all entries of the inverse are integers, then $k = 7.5 + \frac{1}{2n}$, where n is an odd integer. The converse is true as well: If k is chosen in this way, then the entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ will be integers.

24. Compare with Example 5.

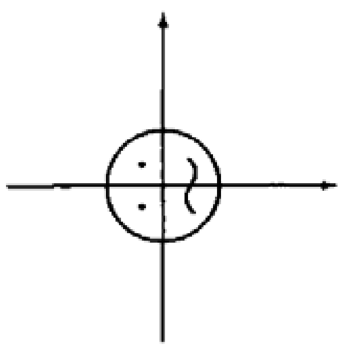

Figure 2.5: for Problem 2.1.24.

25. The matrix represents a scaling by the factor of 2. (See Figure 2.6.)

26. This matrix represents a reflection about the line $x_2 = x_1$.

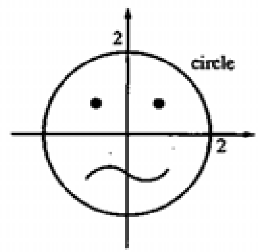

Figure 2.6: for Problem 2.1.25.

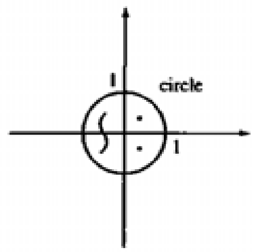

Figure 2.7: for Problem 2.1.26.

27. This matrix represents a reflection about the $\vec{e}_1$ axis. (See Figure 2.8.)

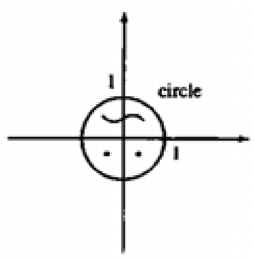

Figure 2.8: for Problem 2.1.27.

28. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged.

29. This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.10.)

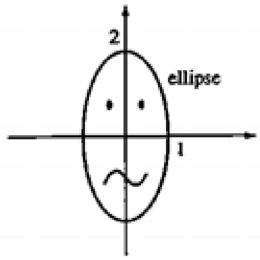

Figure 2.9: for Problem 2.1.28.

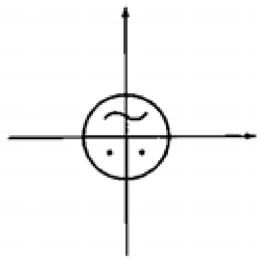

Figure 2.10: for Problem 2.1.29.

30. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the $\vec{e}_2$ axis.

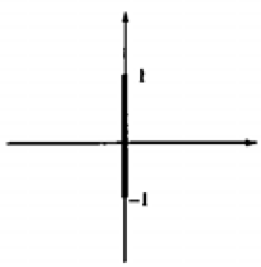

Figure 2.11: for Problem 2.1.30.

42. a. See Figure 2.16.

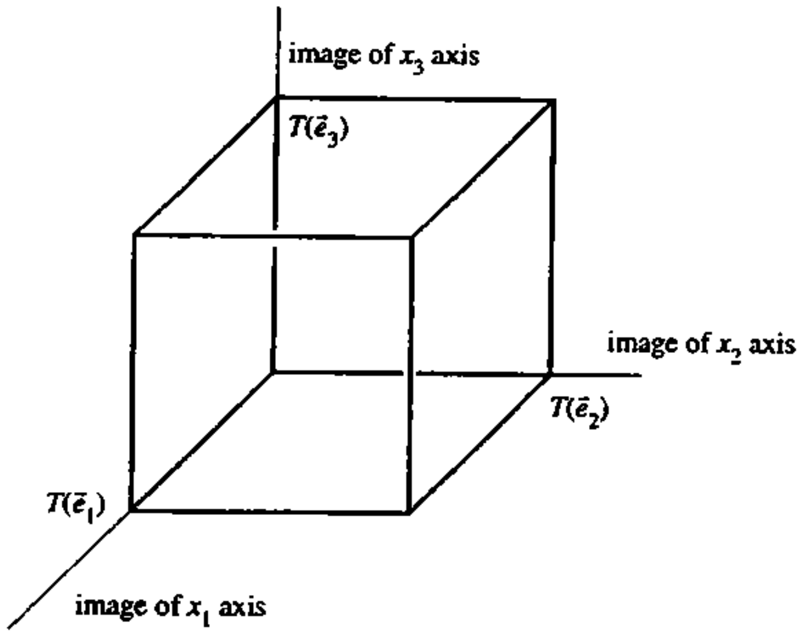

Figure 2.16: for Problem 2.1.42.

b. The image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

c. Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\left| \begin{matrix} -\frac{1}{2}x_1 & + & x_2 & & = 0 \\ -\frac{1}{2}x_1 & + & & x_3 & = 0 \end{matrix} \right|$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$, where t is an arbitrary real number.

For example, for $t = \frac{1}{2}$, we find the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ considered in part b.These points are on the line through the origin and the observer's eye.

52. a. $A\vec{x} = \begin{bmatrix} 420 \\ 2100 \end{bmatrix}$, which is the total value of our money in terms of C$ and R.

b. From Exercise 13, we test the value $ad - bc$ and find it to be zero. Thus A is not invertible. To determine when A is consistent, we begin to compute rref $\left[A \vdots \vec{b}\right]$:

$$\begin{bmatrix} 1 & \frac{1}{5} & \vdots & b_1 \\ 5 & 1 & \vdots & b_2 \end{bmatrix} \begin{matrix} \\ -5I \end{matrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{5} & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 - 5b_1 \end{bmatrix}.$$

Thus, the system is consistent only when $b_2 = 5b_1$. This makes sense, since b_2 is the total value of our money in terms of Rand, while b_1 is the value in terms of Canadian dollars. Consider the example in part a.

If the system $A\vec{x} = \vec{b}$ is consistent, then there will be infinitely many solutions $\vec{x}$, representing various compositions of our portfolio in terms of Rand and Canadian dollars, all representing the same total value.

44. $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

34. As in Exercise 33, we find $T(\vec{e}_1)$ and $T(\vec{e}_2)$; then by Fact 2.1.2, $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$.

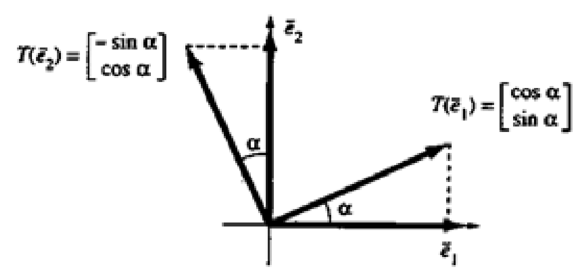

Figure 2.13: for Problem 2.1.34.

Therefore, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.