4. a. For column vectors $\vec{v}, \vec{w}$, we have $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \text{trace}(\vec{v} \cdot \vec{w}) = \vec{v} \cdot \vec{w}$, the dot product.

b. For row vectors $\vec{v}, \vec{w}$, the ijth entry of $\vec{v}^T \vec{w}$ is $v_i w_j$, so that $\langle \vec{v}, \vec{w} \rangle = \text{trace}(\vec{v}^T \vec{w}) = \sum_{i=1}^{m} v_i w_i = \vec{v} \cdot \vec{w}$, again the dot product.

10. A function $g(t) = a + bt + ct^2$ is orthogonal to $f(t) = t$ if

$$\langle f, g \rangle = \int_{-1}^{1} (at + bt^2 + ct^3)\, dt = [\tfrac{a}{2}t^2 + \tfrac{b}{3}t^3 + \tfrac{c}{4}t^4]_{-1}^{1} = \frac{2}{3}b = 0, \text{ that is, if } b = 0.$$

Thus, the functions 1 and t^2 form a basis of the space of all functions in P_2 orthogonal to $f(t) = t$. To find an *orthonormal* basis $g_1(t), g_2(t)$, we apply Gram-Schmidt. Now $\|1\| = \frac{1}{2}\int_{-1}^{1} 1\, dt = 1$, so that we can let $g_1(t) = 1$. Then $g_2(t) = \frac{t^2 - \langle 1, t^2 \rangle 1}{\|t^2 - \langle 1, t^2 \rangle 1\|} = \frac{t^2 - \frac{1}{3}}{\|t^2 - \frac{1}{3}\|} = \frac{\sqrt{5}}{2}(3t^2 - 1)$

Answer: $g_1(t) = 1, g_2(t) = \frac{\sqrt{5}}{2}(3t^2 - 1)$

16. a. We start with the standard basis $1, t$ and use the Gram-Schmidt process to construct an *orthonormal* basis $g_1(t), g_2(t)$.

$\|1\| = \sqrt{\int_0^1 dt} = 1$, so that we can let $g_1(t) = 1$. Then $g_2(t) = \frac{t - \langle 1, t \rangle 1}{\|t - \langle 1, t \rangle 1\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = \sqrt{3}(2t - 1)$.

Summary: $g_1(t) = 1$ and $g_2(t) = \sqrt{3}(2t - 1)$ is an orthonormal basis.

b. We are looking for $\text{proj}_{P_1}(t^2) = \langle g_1(t), t^2 \rangle g_1(t) + \langle g_2(t), t^2 \rangle g_2(t)$, by Fact 5.5.3.

We find that $\langle g_1(t), t^2 \rangle = \int_0^1 t^2\, dt = \frac{1}{3}$ and $\langle g_2(t), t^2 \rangle = \sqrt{3}\int_0^1 (2t^3 - t^2)\, dt = \frac{\sqrt{3}}{6}$, so that $\text{proj}_{P_1} t^2 = \frac{1}{3} + \frac{1}{2}(2t - 1) = t - \frac{1}{6}$. See Figure 5.21.

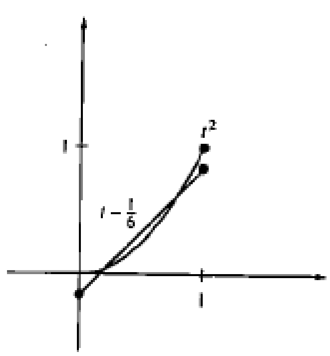

Figure 5.21: for Problem 5.5.16b.

20. a. $\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = [x_1 x_2] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 + 2x_2 = 0$ when $x_1 = -2x_2$. This is the line spanned by vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

b. Since vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are orthogonal, we merely have to multiply each of them with the reciprocal of its norm. Now $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 = [1 \quad 0] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$, so that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a unit vector, and $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\|^2 = [-2 \quad 1] \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4$, so that $\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = 2$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$ is an orthonormal basis.

24. a. $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle = 0 + 8 = 8$

b. $\|g+h\| = \sqrt{\langle g+h, g+h \rangle} = \sqrt{\langle g, g \rangle + 2\langle g, h \rangle + \langle h, h \rangle} = \sqrt{1+6+50} = \sqrt{57}$

c. Since $\langle f, g \rangle = 0$, $\|g\| = 1$, and $\|f\| = 2$, we know that $\frac{f}{2}, g$ is an orthonormal basis of span (f, g).

Now $\text{proj}_E h = \left\langle \frac{f}{2}, h \right\rangle \frac{f}{2} + \langle g, h \rangle g = \frac{1}{4}\langle f, h \rangle f + \langle g, h \rangle g = 2f + 3g$.

d. From part c we know that $\frac{1}{2} f$, g are orthonormal, so we apply Fact 5.2.1 to obtain the third polynomial in an orthonormal basis of span(f, g, h):

$$\frac{h - \text{proj}_E h}{\|h - \text{proj}_E h\|} = \frac{h-2f-3g}{\|h-2f-3g\|} = \frac{h-2f-3g}{5} = -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

Orthonormal basis: $\frac{1}{2}f, g, -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$

22. Apply the Cauchy-Schwarz inequality to $f(t)$ and $g(t) = 1$; note that $\|g\| = 1$:

$$|\langle f, g \rangle| \leq \|f\|\|g\| = \|f\| \text{ or } \langle f, g \rangle^2 \leq \|f\|^2 \text{ or } \left(\int_0^1 f(t)\, dt \right)^2 \leq \int_0^1 (f(t))^2\, dt.$$