28. T is linear, since $T(f(t)+g(t)) = f(2t)+g(2t)-f(t)-g(t) = f(2t)-f(t)+g(2t)-g(t) = T(f(t))+T(g(t))$, and $T(kf(t)) = kf(2t)-kf(t) = k(f(2t)-f(t)) = kT(f(t))$.

T is not an isomorphism, however, since $T(3) = 3-3 = 0$.

34. Linear, since $T((x_0, x_1, x_2, \ldots) + (y_0, y_1, y_2, \ldots)) = T(x_0+y_0, x_1+y_1, x_2+y_2, \ldots) =$

$(0, x_0+y_0, x_1+y_1, x_2+y_2, \ldots)$ equals

$T(x_0, x_1, x_2, \ldots) + T(y_0, y_1, y_2, \ldots) = (0, x_0, x_1, x_2, \ldots) + (0, y_0, y_1, y_2, \ldots) =$

$(0, x_0+y_0, x_1+y_1, x_2+y_2, \ldots)$, and

$T(k(x_0, x_1, x_2, \ldots)) = T(kx_0, kx_1, kx_2, \ldots) = (0, kx_0, kx_1, kx_2, \ldots)$ equals

$kT(x_0, x_1, x_2, \ldots) = k(0, x_0, x_1, x_2, \ldots) = (0, kx_0, kx_1, kx_2, \ldots)$.

No, T isn't an isomorphism, since $(1, 0, 0, 0, \ldots)$ isn't in $\mathrm{im}(T)$.

25. Linear, since $T(f+g) = (f+g)'' + 4(f+g)' = f'' + g'' + 4f' + 4g'$ equals

$T(f) + T(g) = f'' + 4f' + g'' + 4g'$, and $T(kf) = (kf)'' + 4(kf)' = kf'' + 4kf'$ equals

$kT(f) = k(f'' + 4f') = kf'' + 4kf'$. No, it isn't an isomorphism, since the constant function $f(x) = 1$ is in $\ker(T)$.

39. Linear; the proof is analogous to Exercise 25.

No, T isn't an isomorphism, since the kernel of T is two-dimensional, by Fact 4.1.7.

40. Same answer as in Exercise 39.

66. The kernel of T consists of all smooth functions $f(t)$ such that

$T(f(t)) = f(t) - f'(t) = 0$, or $f'(t) = f(t)$. As you may recall from a discussion of exponential functions in calculus, those are the functions of the form $f(t) = Ce^t$, where C is a constant. Thus the nullity of T is 1.

78. a. Check all the conditions in Definition 4.1.1. A basis is 2.

b. $T(x \oplus y) = T(xy) = \ln(xy) = \ln(x) + \ln(y) = T(x) + T(y)$ and

$T(k \odot x) = T(x^k) = \ln(x^k) = k\ln(x) = kT(x)$.

The inverse of T is $L(y) = e^y$, so that T is indeed an isomorphism.

6. Using Fact 9.3.13, $f(t) = e^{2t}\int e^{-2t}e^{2t}\,dt = e^{2t}\int dt = e^{2t}(t+C)$, where C is an arbitrary constant.