

6. $\frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is an orthonormal eigenbasis.

12. a. $E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $E_{-1} = (E_1)^\perp$. An orthonormal eigenbasis is $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

b. Use Fact 7.4.1: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c. $A = SBS^{-1} = \begin{bmatrix} -0.6 & 0 & 0.8 \\ 0 & -1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix}$, where $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{bmatrix}$.

16. a. $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\text{tr}(A) = 5$.

b. $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c. $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

24. Note that A is symmetric and orthogonal, so that the eigenvalues are 1 and -1 (see Exercise 23).

$E_1 = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$ and $E_{-1} = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right)$, so that

$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ is an orthonormal eigenbasis.

28. For $\lambda \neq 0$

$$\begin{aligned}
 f_A(\lambda) &= \det \left[\begin{array}{cccc|c} -\lambda & & & & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ 0 & & & -\lambda & 1 \\ \hline 1 & 1 & \cdots & 1 & 1-\lambda \end{array} \right] = \frac{1}{\lambda} \det \left[\begin{array}{cccc|c} -\lambda & & & & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & & & -\lambda & 1 \\ \hline \lambda & \lambda & \cdots & \lambda & \lambda - \lambda^2 \end{array} \right] \\
 &= \frac{1}{\lambda} \det \left[\begin{array}{cccc|c} -\lambda & & & & 1 \\ & -\lambda & & & 1 \\ & & \ddots & & \vdots \\ & & & -\lambda & 1 \\ \hline 0 & 0 & \cdots & 0 & -\lambda^2 + \lambda + 12 \end{array} \right] \\
 &= -\lambda^{11}(\lambda^2 - \lambda - 12) = -\lambda^{11}(\lambda - 4)(\lambda + 3)
 \end{aligned}$$

Eigenvalues are 0 (with multiplicity 11), 4 and -3 .

Eigenvalues for 0 are $\vec{e}_1 - \vec{e}_i (i = 2, \dots, 12)$,

$$E_4 = \text{span} \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 4 \end{array} \right] \text{ (12 ones)}, \quad E_{-3} = \text{span} \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ -3 \end{array} \right] \text{ (12 ones)}$$

so

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

diagonalizes A , and $D = S^{-1}AS$ will have all zeros as entries except $d_{12, 12} = 4$ and $d_{13, 13} = -3$.

26. Since J_n is both orthogonal and symmetric, the eigenvalues are 1 and -1 . If n is even, then both have multiplicity $\frac{n}{2}$ (as in Exercise 24). If n is odd, then the multiplicities are $\frac{n+1}{2}$ for 1 and $\frac{n-1}{2}$ for -1 (as in Exercise 25). One way to see this is to observe that $\text{tr}(J_n)$ is 0 for even n , and 1 for odd n (recall that the trace is the sum of the eigenvalues).