6. $\lambda_{1,2} = 0.8 \pm (0.6)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.36} = 1$ and $\vec{0}$ is not a stable equilibrium.

16. $\lambda_{1,2} = \frac{2 \pm \sqrt{1+30k}}{10}$ so $|2 \pm \sqrt{1+30k}|$ must be less than 10. $\lambda_{1,2}$ are *real* if $k \geq -\frac{1}{30}$. In this case it is required that $2 + \sqrt{1+30k} < 10$ and $-10 < 2 - \sqrt{1+30k}$, which means that $\sqrt{1+30k} < 8$ or $k < \frac{21}{10}$.

    $\lambda_{1,2}$ are *complex* if $k < -\frac{1}{30}$. Here it is required that $4 + (-1 - 30k) < 100$ or $k > -\frac{97}{30}$. Overall, $\vec{0}$ is a stable equilibrium if $-\frac{97}{30} < k < \frac{21}{10}$.

22. $\lambda_{1,2} = -2 \pm 3i, r = \sqrt{13}, \theta \approx 2.16$ (in second quadrant)

$$[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & -5 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ so } \vec{x}(t) = \sqrt{13}^t \begin{bmatrix} -5\sin(\theta t) \\ \cos(\theta t) - 3\sin(\theta t) \end{bmatrix}, \text{ where } \theta \approx 2.16.$$

Spirals outwards, as in Figure 7.39.

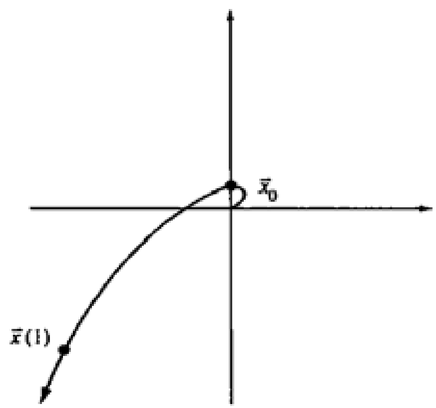


Figure 7.39: for Problem 7.6.22.

34. a. If $|\det A| = |\lambda_1 \lambda_2 \cdots \lambda_n| = |\lambda_1 \lambda_2| \cdots |\lambda_n| > 1$ then at least one eigenvalue is greater than one in modulus and the zero state fails to be stable.

    b. If $|\det A| = |\lambda_1||\lambda_2| \cdots |\lambda_n| < 1$ we cannot conclude anything about the stability of $\vec{0}$.

    $|2||0.1| < 1$ and $|0.2||0.1| < 1$ but in the first case we would not have stability, in the second case we would.

44. T; Note that $S^{-1}AS = D$, so that $D^4 = S^{-1}A^4S = S^{-1}0S = 0$, and therefore $D = 0$ (since $D$ is diagonal) and $A = SDS^{-1} = 0$.

45. T; There is an eigenbasis $\vec{v}_1, \ldots, \vec{v}_n$, and we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. The vectors $c_i\vec{v}_i$ are either eigenvectors or zero.

46. T; If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$, then $AB\vec{v} = \alpha\beta\vec{v}$.

47. T, by Fact 7.3.6a

48. F; Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

38. a. $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.

$I_n - A$ is invertible since 1 is not an eigenvalue of $A$. Therefore, $\vec{v} = (I_n - A)^{-1}\,\vec{b}$ is the only solution.

b. Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium $\vec{v}$.

Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$, so that $\vec{y}(t) = A^t\vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$.

$\lim_{t\to\infty} \vec{x}(t) = \vec{v}$ for all $\vec{x}_0$ if $\lim_{t\to\infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of $A$ is less than 1.

42. a. $x(t+1) = x(t) - ky(t)$

$y(t+1) = kx(t) + y(t) = kx(t) + (1 - k^2)y(t)$ so $\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

b. $f_A(\lambda) = \lambda^2 - (2 - k^2)\lambda + 1 = 0$

The discriminant is $(2 - k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2 - 4)$, which is negative if $k$ is a small positive number ($k < 2$). Therefore, the eigenvalues are complex. By Fact 7.6.4 the trajectory will be an ellipse, since $\det(A) = 1$.