8. By Fact 7.5.1, $(\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^3$, i.e.

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\phi$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$.

Equating real and imaginary parts, we get

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

12. We will use the facts:

i) $\overline{z+w} = \bar{z} + \bar{w}$ and

ii) $(\overline{z^n}) = \bar{z}^n$

These are easy to check. Assume λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ where the coefficients a_i are real. Since λ_0 is a root of f, we have $a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0 = 0$.

Taking the conjugate of both sides we get $\overline{a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0} = \bar{0}$ so by fact i), and factoring the real constants we get $a_n\overline{\lambda_0^n} + a_{n-1}\overline{\lambda_0^{n-1}} + \cdots + a_1\overline{\lambda_0} + a_0 = 0$.

Now, by fact ii), $a_n(\overline{\lambda}_0)^n + a_{n-1}(\overline{\lambda}_0)^{n-1} + \cdots + a_1\overline{\lambda}_0 + a_0 = 0$, i.e. $\overline{\lambda}_0$ is also a root of f, as claimed.

24. $f_A(\lambda) = -\lambda^3 + 3\lambda^2 - 7\lambda + 5$ so $\lambda_1 = 1, \lambda_{2,3} = 1 \pm 2i$. (See Exercise 11.)

32. a. $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$ so $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$.

Note that A is a regular transition matrix.

b. By Exercise 30, $\lim_{t\to\infty}(A^t) = [\vec{v}\,\vec{v}\,\vec{v}]$, where $\vec{v}$ is the unique eigenvector of A with eigenvalue 1 and column sum 1. We find that $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$.

Now $\lim_{t\to\infty}\vec{x}(t) = \lim_{t\to\infty}(A^t\vec{x}_0) = \left(\lim_{t\to\infty} A^t\right)\vec{x}_0 = [\vec{v}\,\vec{v}\,\vec{v}]\vec{x}_0 = \vec{v}$, since the components of $\vec{x}_0$ add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

38. a. $C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, C_4^4 = I_4$, then $C_4^{4+k} = C_4^k$.

Figure 7.31 illustrates how C_4 acts on the basis vectors $\vec{e}_i$.

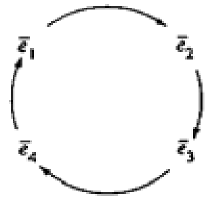

Figure 7.31: for Problem 7.5.38a.

b. The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i$, and $\lambda_4 = -i$, and for each eigenvalue $\lambda_k, \vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$ is an associated eigenvector.

c. $M = aI_4 + bC_4 + cC_4^2 + dC_4^3$

If $\vec{v}$ is an eigenvector of C_4 with eigenvalue λ, then $M\vec{v} = a\vec{v} + b\lambda\vec{v} + c\lambda^2\vec{v} + d\lambda^3\vec{v} = (a + b\lambda + c\lambda^2 + d\lambda^3)\vec{v}$, so that $\vec{v}$ is an eigenvector of M as well, with eigenvalue $a + b\lambda + c\lambda^2 + d\lambda^3$.

The eigenbasis for C_4 we found in part b is an eigenbasis for all circulant 4×4 matrices.

50. The eigenvalues are 0, 0, 1. Since the kernel is always two-dimensional, with basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, the matrix is diagonalizable for all values of constant a.

36. a. The entries in the first row are age-specific birth rates and the entries just below the diagonal are age-specific survival rates. For example, the entry 1.6 in the first row tells us that during the next 15 years the people who are 15–30 years old today will *on average* have 1.6 children (3.2 per couple) who will survive to the next census. The entry 0.53 tells us that 53% of those in the age group 45–60 today will still be alive in 15 years (they will then be in the age group 60–75).

b. Using technology, we find the largest eigenvalue $\lambda_1 = 1.908$ with associated eigenvector

$$\vec{v}_1 \approx \begin{bmatrix} 0.574 \\ 0.247 \\ 0.115 \\ 0.047 \\ 0.014 \\ 0.002 \end{bmatrix}.$$

The components of $\vec{v}_1$ give the distribution of the population among the age groups in the long run, assuming that current trends continue. λ_1 gives the factor by which the population will grow in the long run in a period of 15 years; this translates to an annual growth factor of $\sqrt[15]{1.908} \approx 1.044$, or an annual growth of about 4.4%.