

14. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

If we let $S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

30. First we observe that all the eigenspaces of $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ are one-dimensional, regard-

less of the value of a , since $\text{rref}(A - \lambda I_3)$ is of the form $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$ for all λ . Thus A is diagonalizable if and only if there are three distinct real eigenvalues. The characteristic polynomial of A is $-\lambda^3 + 3\lambda + a$. Thus the eigenvalues of A are the solutions of the equation $\lambda^3 - 3\lambda = a$. See Figure 7.24 with the function $f(\lambda) = \lambda^3 - 3\lambda$; using calculus, we find the local maximum $f(-1) = 2$ and the local minimum $f(1) = -2$. To count the distinct eigenvalues of A , we have to examine how many times the horizontal line $y = a$ intersects the graph of $f(\lambda)$. The answer is three if $|a| < 2$, two if $a = \pm 2$, and one if $|a| > 2$. Thus A is diagonalizable if and only if $|a| < 2$, that is, $-2 < a < 2$.

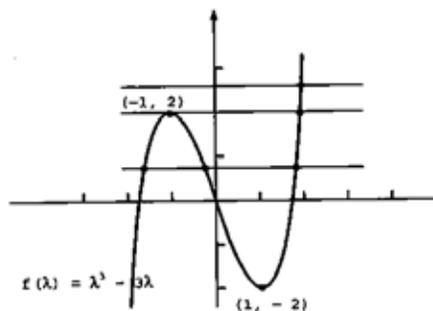


Figure 7.24: for Problem 7.4.30.

36. Yes. The matrices $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, by Algorithm 7.4.4. Thus $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, by parts b and c of Fact 3.4.6.
48. Apply T to the standard basis: $T(1) = 1$, $T(x) = 2x$, and $T(x^2) = (2x)^2 = 4x^2$. This gives the eigenvalues 1, 2, and 4, with corresponding eigenfunctions $1, x, x^2$. Yes, T is diagonalizable, since the standard basis is an eigenbasis for T .
54. Note that $A^2 = 0$, but $B^2 \neq 0$. Since A^2 fails to be similar to B^2 , matrix A isn't similar to B (see Example 7 of Section 3.4).

58. Let $B_i = A - \lambda_i I_n$; note that B_i and B_j commute for any two indices i and j . If \vec{v} is an eigenvector of A with eigenvalue λ_i , then $B_i \vec{v} = \vec{0}$ and $B_1 B_2 \dots B_i \dots B_m \vec{v} = B_1 \dots B_{i-1} B_{i+1} \dots B_m B_i \vec{v} = \vec{0}$. Since A is diagonalizable, any vector \vec{x} in \mathbb{R}^n can be written as a linear combination of eigenvectors, so that $B_1 B_2 \dots B_m \vec{x} = \vec{0}$ and therefore $B_1 B_2 \dots B_m = 0$, as claimed.
56. The hint shows that matrix $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ is similar to $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$; thus matrices M and N have the same characteristic polynomial, by Fact 7.3.6a. Now $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_n & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_n) = (-\lambda)^n f_{AB}(\lambda)$. To understand the second equality, consider Fact 6.1.8. Likewise, $f_N(\lambda) = (-\lambda)^n f_{BA}(\lambda)$. It follows that $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^n f_{BA}(\lambda)$ and therefore $f_{AB}(\lambda) = f_{BA}(\lambda)$, as claimed.