

10.  $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0, E_1 = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_0 = \text{span} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

No eigenbasis

16.  $\lambda_1 = 0$  (no other real eigenvalues), with eigenvector  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

No real eigenbasis

20. For  $\lambda_1 = 1, E_1 = \ker \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  so if  $a = 0$  then  $E_1$  is 2-dimensional, otherwise it is 1-dimensional.

For  $\lambda_2 = 2, E_2 = \ker \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$  so  $E_2$  is 1-dimensional.

Hence, there is an eigenbasis if  $a = 0$ .

28. Since  $J_n(k)$  is triangular, its eigenvalues are its diagonal entries, hence its only eigenvalue is  $k$ . Moreover,

$$E_k = \ker(J_n(k) - kI_n) = \ker \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} = \text{span}(\vec{e}_1).$$

The geometric multiplicity of  $k$  is 1 while its algebraic multiplicity is  $n$ .

42.  $C(t+1) = 0.8C(t) + 10$  so if  $A \begin{bmatrix} C(t) \\ 1 \end{bmatrix} = \begin{bmatrix} C(t+1) \\ 1 \end{bmatrix}, A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix}$ .  $A$  has eigenvectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 50 \\ 1 \end{bmatrix}$  corresponding to  $\lambda_1 = 0.8$  and  $\lambda_2 = 1$ . Since  $\begin{bmatrix} C(0) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -50 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 50 \\ 1 \end{bmatrix}$ , we have  $C(t) = -50(0.9)^t + 50$ , hence in the long run, there will be 50 spectators. The graph of  $C(t)$  looks similar to the graph in Figure 7.21.

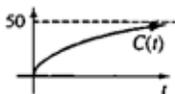


Figure 7.21: for Problem 7.3.42.

38. Note that  $f_A(0) = \det(A - 0I_3) = \det(A) = 1$ .

Since  $\lim_{\lambda \rightarrow \infty} f_A(\lambda) = -\infty$ , the polynomial  $f_A(\lambda)$  must have a positive root  $\lambda_0$ , by the Intermediate Value Theorem. In other words, the matrix  $A$  will have a positive eigenvalue  $\lambda_0$ . Since  $A$  is orthogonal, this eigenvalue  $\lambda_0$  will be 1, by Fact 7.1.2. This means that there is a nonzero vector  $\vec{v}$  in  $\mathbb{R}^3$  such that  $A\vec{v} = 1\vec{v} = \vec{v}$ , as claimed. See Figure 7.19.

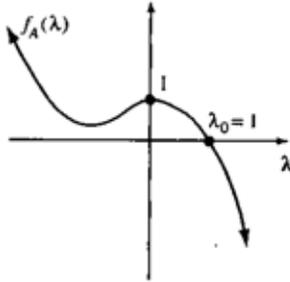


Figure 7.19: for Problem 7.3.38.

46. a.  $T_1(t+1) = 0.6T_1(t) + 0.1T_2(t) + 20$

$$T_2(t+1) = 0.1T_1(t) + 0.6T_2(t) + 0.1T_3(t) + 20$$

$$T_3(t+1) = 0.1T_2(t) + 0.6T_3(t) + 40$$

$$\text{so } A = \begin{bmatrix} 0.6 & 0.1 & 0 \\ 0.1 & 0.6 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 20 \\ 20 \\ 40 \end{bmatrix}$$

b.  $B = \begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$

c.  $\vec{y}(10) = B^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 70.86 \\ 93.95 \\ 120.56 \\ 1 \end{bmatrix}$

$$\vec{y}(30) = B^{30} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 74.989 \\ 99.985 \\ 124.989 \\ 1 \end{bmatrix}$$

$$\vec{y}(t) \text{ seems to approach } \begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix} \text{ as } t \rightarrow \infty$$

d. The eigenvalues of  $A$  are  $\lambda_1 \approx 0.45858$ ,  $\lambda_2 = 0.6$ ,  $\lambda_3 \approx 0.74142$  so the eigenvalues of  $B$  are  $\lambda_1 \approx 0.45858$ ,  $\lambda_2 = 0.6$ ,  $\lambda_3 \approx 0.74142$ ,  $\lambda_4 = 1$ .

If  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are eigenvectors of  $A$  (with  $A\vec{v}_i = \lambda_i\vec{v}_i$ ), then  $\begin{bmatrix} \vec{v}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} \vec{v}_3 \\ 0 \end{bmatrix}$  are

corresponding eigenvectors of  $B$ . Furthermore,  $\begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix}$  is an eigenvector of  $B$  with

eigenvalue 1. Since  $\lambda_1, \lambda_2, \lambda_3$  are all less than 1,  $\lim_{t \rightarrow \infty} \vec{x}(t) = \begin{bmatrix} 75 \\ 100 \\ 125 \\ 1 \end{bmatrix}$ , as in Exercise 45.

32. Recall that a matrix and its transpose have the same rank (Fact 5.3.9c). The geometric multiplicity of  $\lambda$  as an eigenvalue of  $A$  is  $\dim(\ker(A - \lambda I_n)) = n - \text{rank}(A - \lambda I_n)$ .

The geometric multiplicity of  $\lambda$  as an eigenvalue of  $A^T$  is  $\dim(\ker(A^T - \lambda I_n))$

$$= \dim(\ker(A - \lambda I_n)^T) = n - \text{rank}(A - \lambda I_n)^T = n - \text{rank}(A - \lambda I_n).$$

We can see that the two multiplicities are the same.