8. This matrix fails to be invertible, since the $\det(A) = 0$.

18. $\det\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix} = 30 + 21k - 18k^2 = -3(k-2)(6k+5)$. So k cannot be 2 or $-\frac{5}{6}$.

40. We repeatedly expand down the first column, finding the determinant to be

$5(4)(3)\det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = -120$.

43. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\det(-A) = \det\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = ad - bc = \det(A)$.
Use Sarrus' rule to see that $\det(-A) = -\det(A)$ for a 3×3 matrix. We may conjecture

that $\det(-A) = \det(A)$ for an $n \times n$ matrix with even n, and $\det(-A) = -\det(A)$ when n is odd; in both cases we can write $\det(-A) = (-1)^n \det(A)$. A proof of this conjecture can be based upon the following rule: If a square matrix B is obtained from matrix A by multiplying all the entries in the i^{th} row of A by a scalar k, then $\det(B) = k\det(A)$. We can show this by expansion along the i^{th} row:

$$\det(B) = \sum_{j=1}^{n}(-1)^{i+j} b_{ij}\det(B_{ij}) = \sum_{j=1}^{n}(-1)^{i+j} k a_{ij}\det(A_{ij})$$

$$= k\sum_{j=1}^{n}(-1)^{i+j} a_{ij}\det(A_{ij}) = k\det(A).$$

We can obtain $-A$ by multiplying all the entries in the first row of A by $k = -1$, then all the entries in the second row, and so forth down to the n^{th} row; each time the determinant gets multiplied by $k = -1$. Thus, $\det(-A) = k^n \det(A) = (-1)^n \det(A)$, as claimed.

44. $\det(kA) = k^n \det(A)$

The argument is analogous to the one in Exercise 43.

Solution for Problem A

Part A:

All determinants evaluated using the Laplace Expansion with respect to the first columns:

$a_1 = \det[4] = 4$

$a_2 = \det\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} = 4*\det[4] - 3*\det[3] = 4*4 - 3*3 = 7$

$a_3 = \det\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & 3 \\ 0 & 3 & 4 \end{bmatrix} = 4*\det\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} - 3*\det\begin{bmatrix} 3 & 0 \\ 3 & 4 \end{bmatrix} = 4*a_2 - 3*\det\begin{bmatrix} 3 & 0 \\ 3 & 4 \end{bmatrix}$

$= 4*a_2 - 3*3*\det[4] = 4*a_2 - 9*a_1 = 4*7 - 9*4 = 28 - 36 = -8$

Part B:

Evaluate the determinant of L_n using the Laplace Expansion, with respect to the first column of each matrix:

$$a_n = \det\begin{bmatrix} 4 & 3 & 0 & ... & 0 & 0 & 0 \\ 3 & 4 & 3 & ... & 0 & 0 & 0 \\ 0 & 3 & 4 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 4 & 3 & 0 \\ 0 & 0 & 0 & ... & 3 & 4 & 3 \\ 0 & 0 & 0 & ... & 0 & 3 & 4 \end{bmatrix} = 4*\det\begin{bmatrix} 4 & 3 & ... & 0 & 0 & 0 \\ 3 & 4 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 4 & 3 & 0 \\ 0 & 0 & ... & 3 & 4 & 3 \\ 0 & 0 & ... & 0 & 3 & 4 \end{bmatrix} - 3*\det\begin{bmatrix} 3 & 3 & ... & 0 & 0 & 0 \\ 0 & 4 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 4 & 3 & 0 \\ 0 & 0 & ... & 3 & 4 & 3 \\ 0 & 0 & ... & 0 & 3 & 4 \end{bmatrix}$$

$$= 4a_{n-1} - 3*\det\begin{bmatrix} 3 & 3 & ... & 0 & 0 & 0 \\ 0 & 4 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 4 & 3 & 0 \\ 0 & 0 & ... & 3 & 4 & 3 \\ 0 & 0 & ... & 0 & 3 & 4 \end{bmatrix} = 4a_{n-1} - 3*3*\det\begin{bmatrix} 4 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 4 & 3 & 0 \\ 0 & ... & 3 & 4 & 3 \\ 0 & ... & 0 & 3 & 4 \end{bmatrix} = 4a_{n-1} - 9a_{n-2}$$

Part C:

Use the recursion formula solved for in part B to evaluate the determinant of L_{10}:

$$a_4 = 4*a_3 - 9*a_2 = 4*(-8) - 9*(7) = -95$$
$$a_5 = 4*a_4 - 9*a_3 = 4*(-95) - 9*(-8) = -308$$
$$a_6 = 4*a_5 - 9*a_4 = 4*(-308) - 9*(-95) = -377$$
$$a_7 = 4*a_6 - 9*a_5 = 4*(-377) - 9*(-308) = 1264$$
$$a_8 = 4*a_7 - 9*a_6 = 4*(1264) - 9*(-377) = 8449$$
$$a_9 = 4*a_8 - 9*a_7 = 4*(8449) - 9*(1264) = 22420$$
$$a_{10} = 4*a_9 - 9*a_8 = 4*(22420) - 9*(8449) = 13639$$

42. We first expand across the fourth row to obtain

$3\det\begin{bmatrix}1 & 1 & 0 & 1\\ 0 & 2 & 0 & 0\\ 2 & 3 & 4 & 0\\ 3 & 4 & 5 & 6\end{bmatrix}$, then across the second row to obtain

$$3(2)\det\begin{bmatrix}1 & 0 & 1\\ 2 & 4 & 0\\ 3 & 5 & 6\end{bmatrix} = 6(24 + 10 - 12) = 132.$$

56. a. $d_n = \det\begin{bmatrix}0 & \\ \vdots & M_{n-1}\\ 0 & \\ 1 & \end{bmatrix}$.

Now, if we expand down the first column, we find that $d_n = \det(M_{n-1})$ if n is odd, or $d_n = -\det(M_{n-1})$ if n is even. We can model this by simply saying: $d_n = (-1)^{n-1}d_{n-1}$.

b. $d_1 = 1, d_2 = -1, d_3 = -1, d_4 = 1, d_5 = 1, d_6 = -1, d_7 = -1, d_8 = 1$. We notice the pattern that it keeps switching between -1 and 1 with every other increase. We see that $d_{n+4} = d_n$.

c. By the periodicity in part b, we see that d_{100} will be equal to $d_{100-24(4)} = d_4 = 1$.