

8. Since $\vec{u} \cdot \vec{v} = 4 - 24 + 20 = 0$, the two vectors enclose a right angle.

16. You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors \vec{x} that are orthogonal to \vec{v}_1, \vec{v}_2 , and \vec{v}_3 , then we identify the unit vectors among them.

Finding the vectors \vec{x} with $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$ amounts to solving the system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$$

(we can omit all the coefficients $\frac{1}{2}$).

$$\text{The solutions are of the form } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}.$$

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for \vec{v}_4 :

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

18. a. $\|\vec{x}\|^2 = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ (use the formula for a geometric series, with $a = \frac{1}{4}$), so that $\|\vec{x}\| = \frac{2}{\sqrt{3}} \approx 1.155$.

b. If we let $\vec{u} = (1, 0, 0, \dots)$ and $\vec{v} = (1, \frac{1}{2}, \frac{1}{4}, \dots)$, then

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{1}{\sqrt{3}} = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} (= 30^\circ).$$

c. $\vec{x} = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{n}}, \dots)$ does the job, since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges (a fact discussed in introductory calculus classes).

d. If we let $\vec{v} = (1, 0, 0, \dots)$, $\vec{x} = (1, \frac{1}{2}, \frac{1}{4}, \dots)$ and $\vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \frac{\sqrt{3}}{2} (1, \frac{1}{2}, \frac{1}{4}, \dots)$ then

$$\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v}) \vec{u} = \frac{3}{4} (1, \frac{1}{2}, \frac{1}{4}, \dots).$$

20. On the line L spanned by \vec{x} we want to find the vector $m\vec{x}$ closest to \vec{y} (that is, we want $\|m\vec{x} - \vec{y}\|$ to be minimal). We want $m\vec{x} - \vec{y}$ to be perpendicular to L (that is, to \vec{x}), which means that $\vec{x} \cdot (m\vec{x} - \vec{y}) = 0$ or $m(\vec{x} \cdot \vec{x}) - \vec{x} \cdot \vec{y} = 0$ or $m = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}} \approx \frac{4182.9}{198.53^2} \approx 0.106$.

Recall that the correlation coefficient r is $r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$, so that $m = \frac{\|\vec{y}\|}{\|\vec{x}\|} r$. See Figure 5.3.

28. Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Fact 5.1.5, with $\vec{x} = \vec{e}_1$: $\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3 = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

38. Since \vec{v}_1 and \vec{v}_2 are unit vectors, the condition $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos(\alpha) = \cos(\alpha) = \frac{1}{2}$ implies that \vec{v}_1 and \vec{v}_2 enclose an angle of 60° ($= \frac{\pi}{3}$). The vectors \vec{v}_1 and \vec{v}_3 enclose an angle of 60° as well.

In the case $n = 2$ there are two possible scenarios: either $\vec{v}_2 = \vec{v}_3$, or \vec{v}_2 and \vec{v}_3 enclose an angle of 120° . Therefore, either $\vec{v}_2 \cdot \vec{v}_3 = 1$ or $\vec{v}_2 \cdot \vec{v}_3 = \cos(120^\circ) = -\frac{1}{2}$.

In the case $n = 3$, the vectors \vec{v}_2 and \vec{v}_3 could enclose any angle between 0° (if $\vec{v}_2 = \vec{v}_3$) and 120° , as illustrated in Figure 5.7. We have $-\frac{1}{2} \leq \vec{v}_2 \cdot \vec{v}_3 \leq 1$.

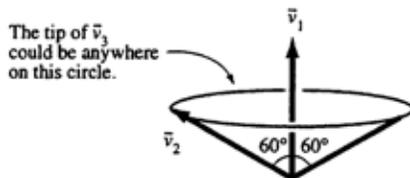


Figure 5.7: for Problem 5.1.38.

For example, consider $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right) \cos \theta \\ \left(\frac{\sqrt{3}}{2}\right) \sin \theta \\ \frac{1}{2} \end{bmatrix}$

Note that $\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{3}{4}\right) \sin \theta + \frac{1}{4}$ could be anything between $-\frac{1}{2}$ (when $\sin \theta = -1$) and 1 (when $\sin \theta = 1$), as claimed.

If n exceeds three, we can consider the orthogonal projection \vec{w} of \vec{v}_3 onto the plane E spanned by \vec{v}_1 and \vec{v}_2 .

Since $\text{proj}_{\vec{v}_1} \vec{w} = (\vec{v}_1 \cdot \vec{w})\vec{v}_1 = \frac{1}{2}\vec{v}_1$, and since $\|\vec{w}\| \leq \|\vec{v}_3\| = 1$, (by Fact 5.1.10), the tip of \vec{w} will be on the line segment in Figure 5.8. Note that θ is between 0° and 120° , so that $\cos \theta$ is between $-\frac{1}{2}$ and 1.

Therefore, $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos \theta$ is between $-\frac{1}{2}$ and 1.

This implies that $\angle(\vec{v}_2, \vec{v}_3)$ is between 0° and 120° as well. To see that all these values are attained, add $(n-3)$ zeros to the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbb{R}^3 given above.

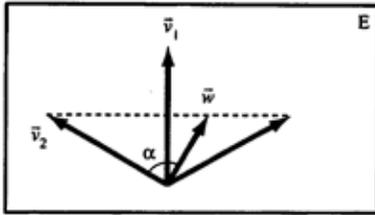


Figure 5.8: for Problem 5.1.38.

14. The horizontal components of \vec{F}_1 and \vec{F}_2 are $-\|\vec{F}_1\| \sin \beta$ and $\|\vec{F}_2\| \sin \alpha$, respectively (the horizontal component of \vec{F}_3 is zero).

Since the system is at rest, the horizontal components must add up to 0, so that $-\|\vec{F}_1\| \sin \beta + \|\vec{F}_2\| \sin \alpha = 0$ or $\|\vec{F}_1\| \sin \beta = \|\vec{F}_2\| \sin \alpha$ or $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin \alpha}{\sin \beta}$.

To find $\frac{\overline{EA}}{\overline{EB}}$, note that $\overline{EA} = \overline{ED} \tan \alpha$ and $\overline{EB} = \overline{ED} \tan \beta$ so that $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan \alpha}{\tan \beta} = \frac{\sin \alpha}{\sin \beta} \cdot \frac{\cos \beta}{\cos \alpha} = \frac{\|\vec{F}_1\| \cos \beta}{\|\vec{F}_2\| \cos \alpha}$. Since α and β are two distinct acute angles, it follows that $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$, so that Leonardo was mistaken.