4. This subset V is a subspace of P_2:

- The neutral element $f(t) = 0$ (for all t) is in V since $\int_0^1 0\,dt = 0$.
- If f and g are in V $\left(\text{so that } \int_0^1 f = \int_0^1 g = 0\right)$ then $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0$, so that $f+g$ is in V.
- If f is in V $\left(\text{so that } \int_0^1 f = 0\right)$ and k is any constant, then $\int_0^1 kf = k\int_0^1 f = 0$, so that kf is in V.

If $f(t) = a + bt + ct^2$ then $\int_0^1 f(t)dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3\right]_0^1 = a + \frac{b}{2} + \frac{c}{3} = 0$ if $a = -\frac{b}{2} - \frac{c}{3}$.

The general element of V is $f(t) = \left(-\frac{b}{2} - \frac{c}{3}\right) + bt + ct^2 = b\left(t - \frac{1}{2}\right) + c\left(t^2 - \frac{1}{3}\right)$, so that $t - \frac{1}{2},\ t^2 - \frac{1}{3}$ is a basis of V.

14. Yes

- $(0, 0, 0, \ldots, 0, \ldots)$ converges to 0.
- If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$.
- If $\lim_{n\to\infty} x_n = 0$ and k is any constant, then $\lim_{n\to\infty}(kx_n) = k\lim_{n\to\infty} x_n = 0$.

42. Let B be a matrix such that $\dim(\ker(B)) = k$. Then, it is required that the columns of A contain only vectors in the kernel of B. Thus, each column of A can be written as: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, where the vectors $\vec{v}_i$ form as basis of the kernel of B. Thus, each of the n columns in A involves k arbitrary constants, and matrix A involves nk arbitrary constants overall. The space of matrices A has dimension nk, where k is an integer in the range $[0, n]$.

50. Using Example 18 as a guide, we first look for solutions of the form $f(x) = e^{kx}$. It is required that $f''(x) + 8f'(x) - 20f(x) = k^2e^{kx} + 8ke^{kx} - 20e^{kx} = 0$ for all x, or $k^2 + 8k - 20 = (k-2)(k+10) = 0$. Thus $k = 2$ or $k = -10$. By Fact 4.1.5, the solutions of the differential equation are of the form $f(x) = c_1e^{2x} + c_2e^{-10x}$, where c_1 and c_2 are arbitrary constants.

52. We have to find constants a and b such that the functions e^{-x} and e^{-5x} are solutions of the differential equation $f''(x) + af'(x) + bf(x) = 0$. Thus it is required that $e^{-x} - ae^{-x} + be^{-x} = 0$, or $1 - a + b = 0$, and also that $25 - 5a + b = 0$. The solution of this system of two equations in two unknowns is $a = 6, b = 5$, so that the desired differential equation is $f''(x) + 6f'(x) + 5f(x) = 0$.

38. We look at similar cases mentioned in Exercise 37, and see that the different possibilities occur when all four entries are different ($\dim(V) = 4$), when exactly two are the same, but the other two are different ($\dim(V) = 6$), when exactly three are the same ($\dim(V) = 10$), when all four are the same ($\dim(V) = 16$) and when two of the terms of B are equal, and the other two diagonal terms of B are also equal, but different from the first pair ($\dim(V) = 8$).

56. Argue indirectly and assume that the space V of infinite sequences is finite-dimensional, with $\dim(V) = n$. According to the solution to Exercise 57, there can be at most n linearly independent elements in V. But here is our contradiction: It is easy to give $n+1$ linearly independent infinite sequences, namely,

$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots, (0, 0, 0, \ldots, 0, 1, 0, \ldots)$; in the last sequence the 1 is in the $(n+1)$th place.