FORTH PRACTICE EXAM SECOND HOURLY **Math 21b, Spring 08**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Show all your work and justify steps.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F** Let A and B be two $n \times n$ matrices. Then A and B are similar if and only if they have the same characteristic polynomials.

Solution:
The shear is a counter example.

2) T **F** Let A be a $n \times n$ matrix. Applying Gauss-Jordan row operations to A does never change the determinant.

Solution:
For example: switching two rows changes the sign.

3) T **F** An orthogonal matrix is symmetric $A^T = A$ or anti symmetric $A^T = -A$.

Solution:
This is already false for rotations in the plane.

4) T **F** The eigenvalues of a matrix A do not change under row reduction.

Solution:
Every invertible matrix reduces to the identity matrix.

5) T **F** The eigenvectors of a matrix A do not change under row reduction.

Solution:
Look at the shear. It has only one eigenvector. After row reduction, it has two different eigenvectors.

6) **T** F If A is the matrix of a reflection at a line in space, then $\det(A+I) = 0$ and $\det(A-I) = 0$.

Solution:
Indeed, every nonzero vector perpendicular to the line is an eigenvector to the eigenvalue -1, and every nonzero vector in the line is an eigenvector to the eigenvalue 1.

7) T **F** Every upper triangular matrix can be diagonalized.

Solution:
The shear is a counter example

8) T **F** There is a recursion $x_{n+1} = ax_n + bx_{n-1}$ for which $x_n > (1.01)^{2^n}$ for all n.

Solution:
Diagonalization of linear systems shows that x_n can only grow exponentially.

9) T **F** The sum of two projections is a projection.

Solution:
The identity is a projection. The sum of two identity transformations is a dilation by a factor 2 and not a projection.

10) **T** F The characteristic polynomial of A is the same as the characteristic polynomial of A^T.

Solution:
Follows from $\det(A) = \det(A^T)$.

11) **T** F For any matrix, we have $\det(A^5) = \det(A)^5$.

Solution:
Yes, $\det(AB) = \det(A)\det(B)$.

12) T **F** The trace of a matrix is equal to the product of the eigenvalues.

Solution:
It is the sum of the eigenvalues.

13) T **F** There is a projection for which the determinant is equal to 2.

Solution:
The eigenvalues are 1 or 0. The determinant can be 1 or 0.

14) **T** F The matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$.

Solution:
Because both have the same trace and determinant, their eigenvalues are the same. Because the eigenvalues are different they can both be diagonalized to the same diagonal matrix.

15) T **F** If two 2×2 matrices A and B have the same trace and determinant, then they are similar.

Solution:
The shear A and the shear A^2 are not conjugated, but they have the same trace and determinant.

16) **T** F If x is a solution to the linear equation $Ax = b$, then x is a least square solution to $Ax = b$.

Solution:
Yes, then the quadratic error is zero.

17) **T** F It is possible that the length of $A^n v$ and the length of $A^{-n} v$ both grow exponentially.

Solution:
This is the case, if one eigenvalues is smaller than 1 and the other is bigger than 1.

18) T F — If an orthogonal matrix Q is symmetric, then Q is diagonal.

Solution:
Take a reflection at a line.

19) T F — If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.

Solution:
Take the case, where A is an orthogonal matrix. Then $A = A1$ is the QR decomposition, but the eigenvalues of A are not necessarily all 1.

20) T F — For every invertible $n \times n$ matrix A, there is a nonzero $n \times n$ matrix B such that AB is the zero matrix.

Solution:
If AB is the zero matrix, then B must be the zero matrix. Otherwise, there is a vector e_i such that Be_i is not the zero vector and then ABe_i is not the zero vector.

	Total

Problem 2) (10 points)

No explanations are necessary for this problem.

a) Which matrices are orthogonal, which matrices are symmetric, which matrices are projections? Check everything which applies. It is not excluded that you have to check several properties for each matrix.

	orthogonal	symmetric	projection	
1)				$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
2)				$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3)				$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
4)				$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) Which of the following assertions are true?

true	false	
		A is similar to B
		A is similar to C
		A is similar to D

		B is similar to C
		B is similar to D
		C is similar to D

Solution:

	orthogonal	symmetric	projection	
a)	X	X		$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
b)	X			$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
c)		X	X	$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
d)		X		$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

true	false	
	X	A is similar to B
	X	A is similar to C
	X	A is similar to D
	X	B is similar to C
	X	B is similar to D
	X	C is similar to D

c) Match the following matrices with sets of eigenvalues. There is a unique match. It is not necessary to compute the eigenvalues to do so. Enter i),ii) or iii) in the boxes.

	$A = \begin{bmatrix} -5 & -9 & -7 \\ 2 & 5 & 2 \\ 4 & 5 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

i) $\{3, 2, 1\}$.
ii) $\{1, 0, 3\}$.
iii) $\{6, 5, 5\}$.

Solution:
c) i), iii),ii). Note that the trace of the matrix is the sum of the eigenvalues.

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A with their algebraic multiplicities.

b) Find the geometric multiplicities of each of the eigenvalues.

c) Find all the eigenvectors.

d) What is $\det(A)$?

Solution:
Look at it as a partitioned matrix:
a) $2+\sqrt{3}, 2-\sqrt{3}, 3, 3, 3+i, 3-i$ are the eigenvalues.
b) The geometric multiplicities are 1 for all eigenvalues. Note that 3 appears twice and has the algebraic multiplicity 2 but the geometric multiplicity is 1.
c) The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i \\ 1 \end{bmatrix} v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

The eigenvector v_3 is the only eigenvector to the eigenvalue 3.
d) The determinant is the product of the eigenvalues which is $\boxed{90}$.

Problem 4) (10 points)

Find the function $y = f(x) = a\cos(\pi x) + b\sin(\pi x)$, which best fits the data

x	y
0	1
1/2	3
1	7

Solution:
We have to find the least square solution to the system of equations

$$\begin{aligned} 1a + 0b &= 1 \\ 0a + 1b &= 3 \\ -1a + 0b &= 7 \end{aligned}$$

which is in matrix form written as $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}.$$

Now $A^T\vec{b} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ and $(A^TA)^{-1}A^T\vec{b}$ is $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$. The best fit is the function $\boxed{f(x) = -3\cos(\pi x) + 3\sin(\pi x)}$.

Problem 5) (10 points)

a) Find all the eigenvalues λ_1, λ_2 and λ_3 of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Find a formula for $\text{tr}(A^n)$.

Solution:
a) The characteristic polynomial is

$$f_A(\lambda) = -\lambda^3 + 2\lambda^2 - 3\lambda + 2 \ .$$

eigenvalues are

$$\lambda_1 = (1 + i\sqrt{7})/2, \lambda_2 = (1 - i\sqrt{7})/2, \lambda_3 = 1$$

One can also look at the matrix as a partitioned matrix. There is a 2×2 matrix in the upper left corner which has the eigenvalues λ_1, λ_2.
b) Because $\text{tr}(A^n) = \lambda_1^n + \lambda_2^n + \lambda_3^n$, we have

$$\text{tr}(A^n) = \frac{(1 + i\sqrt{7})^n}{2^n} + \frac{(1 - i\sqrt{7})^n}{2^n} + 1 \ .$$

Problem 6) (10 points)

a) (5 points) Find an eigenbasis of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

b) (5 points) Do Gram-Schmidt orthogonalization on the basis $\mathcal{B} = \{v_1, v_2, v_3\ \}$ you just got. Write down the QR decomposition of the matrix S which contains the basis $\mathcal{B}$ as column vectors.

Solution:
a) The eigenvalues are $4, 2, 1$. They are all different. The eigenvectors are

$$\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

b) To make a Gram-Schmidt orthogonalization, we use the fact that we can reorder the basis as we want. So, instead of for example $S = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we better take $S = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ which is upper trianguler Gram Schmidt orthonormalization produces the standard basis and the QR decomposition is $S = I_3 S$ because S is already upper triangular. The QR factorization is

$$S = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = QR .$$

Problem 7) (10 points)

a) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 4 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

b) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}$$

Solution:
a) Swap rows to get an upper triangular matrix. You can do that with 4 swaps. The determinant is $\boxed{120}$.
b) This is a partitioned matrix having 2 matrices of size 3×3 in the diagonal. The determinant of A is the product of the determinants of these matrices which is $=3^3 \cdot 4^3 = \boxed{1728}$.

Problem 8) (10 points)

We want to find a formula for the general term x_n in the recursion

$$x_{n+1} = x_n + 3x_{n-1}/4$$

if $x_0 = 0, x_1 = 1$. This is the case of a Fibonacci recursion, in which only 3/4 of the previous generation has kids.

a) Write the recursion in the form $v_{n+1} = Av_n$ for vectors $v_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in the plane.
b) Find the eigenvalues λ_+, λ_- and eigenvectors v_+, v_- of A.
c) Write $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $v_0 = av_+ + bv_-$.
d) Find $v_n = A^n v_0$ and so x_n.

Solution:
a) $A = \begin{bmatrix} 1 & 3/4 \\ 1 & 0 \end{bmatrix}$. b) The characteristic polynomial is $f_A(\lambda) = x^2 - x - 3/4$ which has the roots $\lambda_+ = 3/2$ and $\lambda_- = -1/2$. These are the eigenvalues.
c) The eigenvectors are

$$v_+ = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and

$$v_- = \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = v = v_+/4 - v_-/4$. Therfore $A^n v = \lambda_+^n v_+ + \lambda_-^n v_-$. Written out, this is

$$A^n \langle [\begin{matrix} 1 \\ 0 \end{matrix} \rangle] = \frac{3^n}{4 \cdot 2^n} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{(-1)^n}{4 \cdot 2^n} \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

Therefore, $\boxed{x_n = \frac{3^n - (-1)^n}{2^{n+1}}}$.

Problem 9) (10 points)

Our goal is to find the determinant of

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 \\ 2 & 102 & 2 & 2 & 2 & 2 \\ 3 & 3 & 103 & 3 & 3 & 3 \\ 4 & 4 & 4 & 104 & 4 & 4 \\ 5 & 5 & 5 & 5 & 105 & 5 \\ 6 & 6 & 6 & 6 & 6 & 106 \end{bmatrix}$$

a) The matrix $A - 100I_6$ has an eigenvalue 0. Find its algebraic multiplicity.

b) The matrix A^T has an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the corresponding eigenvalue λ.

c) Why does A also have the same eigenvalue λ?

d) You have found all the eigenvalues of $A - 100I_6$. What are the eigenvalues of A?

e) Find the determinant of A.

Solution:

a) The matrix

$$B = A - 100I_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

has identical columns and is therefore not invertible. It must have an eigenvalue 0. Row reduction gives

$$\text{rref}(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that the image is 1 dimensional and the kernel is 5 dimensional. Because the geometric multiplicity is 5, the algebraic multiplicity is 5 or larger. Because the trace is not zero, the algebraic multiplicity can not be 6. The trace is the sum of the eigenvalues would be 0 if the algebraic multiplicity of 0 would be 6.

b) B^T has an eigenvalue $\lambda = 21$. Because B^T and B have the same eigenvalues (they have the same characteristic polynomial because transposed matrices have the same determinant), also B has the eigenvalue 21.

c) $A = B + 100I_6$ has the eigenvalue $21 + 100$ with algebraic multiplicity 1 and $0 + 100$ with algebraic multiplicity 5.

d) The determinant of A is the product of the eigenvalues, which is $\boxed{100^5 \cdot 121}$.