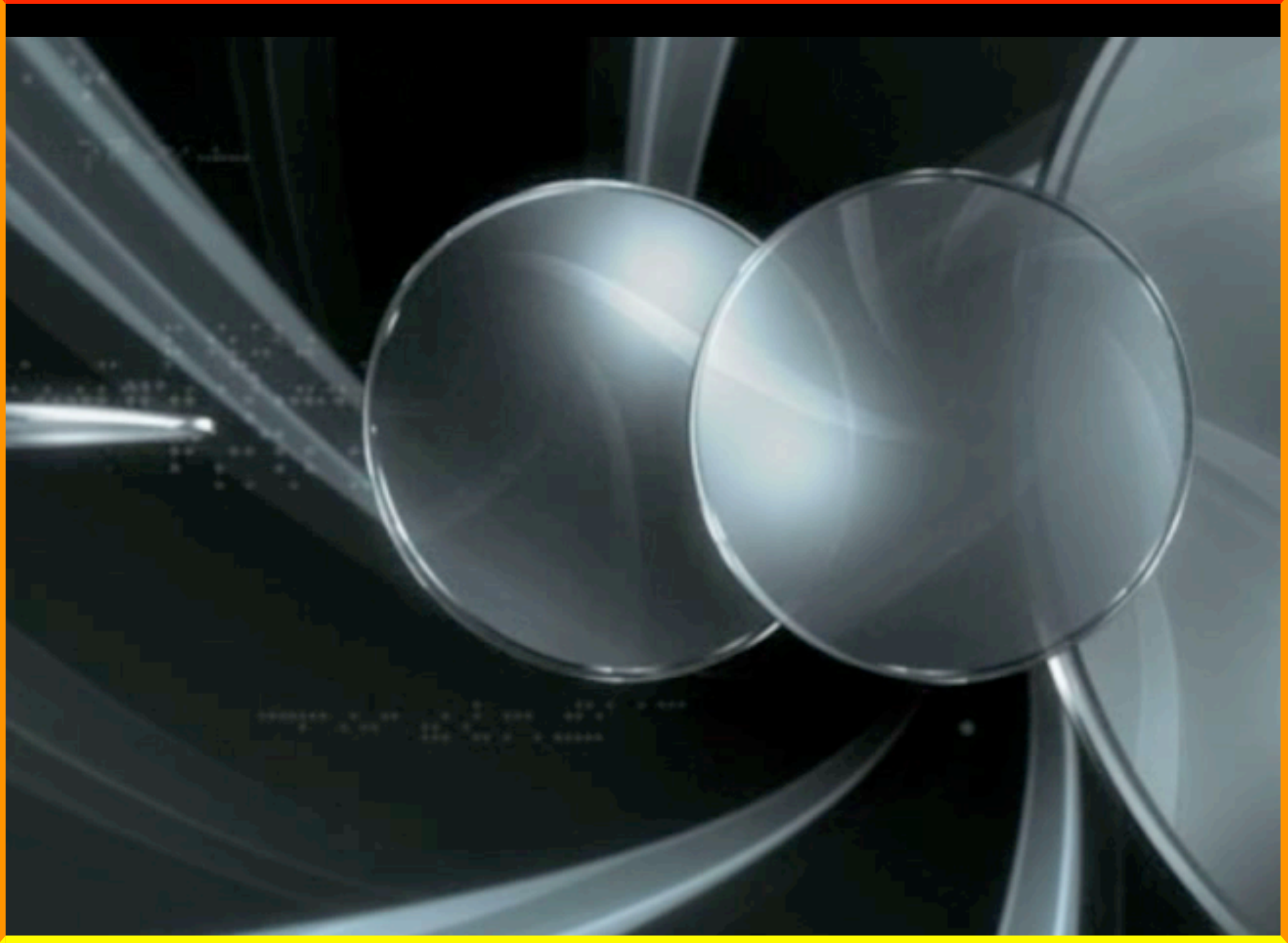
Math21b
Review to second midterm
Spring 2008

Oliver Knill



Orthonormal basis

$\{ w_1, w_2, \ldots, w_n \}$ orthonormal basis

stick this orthonormal basis as columns in the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} /2$$

this gives an orthogonal matrix. It satisfies:

$$A^T A = I$$

Also the following matrix has an orthonormal set of vectors as columns.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} /2$$

we have

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But the matrix A is not an orthogonal matrix.

Orthogonal transformation

$$T(x) = A\,x$$

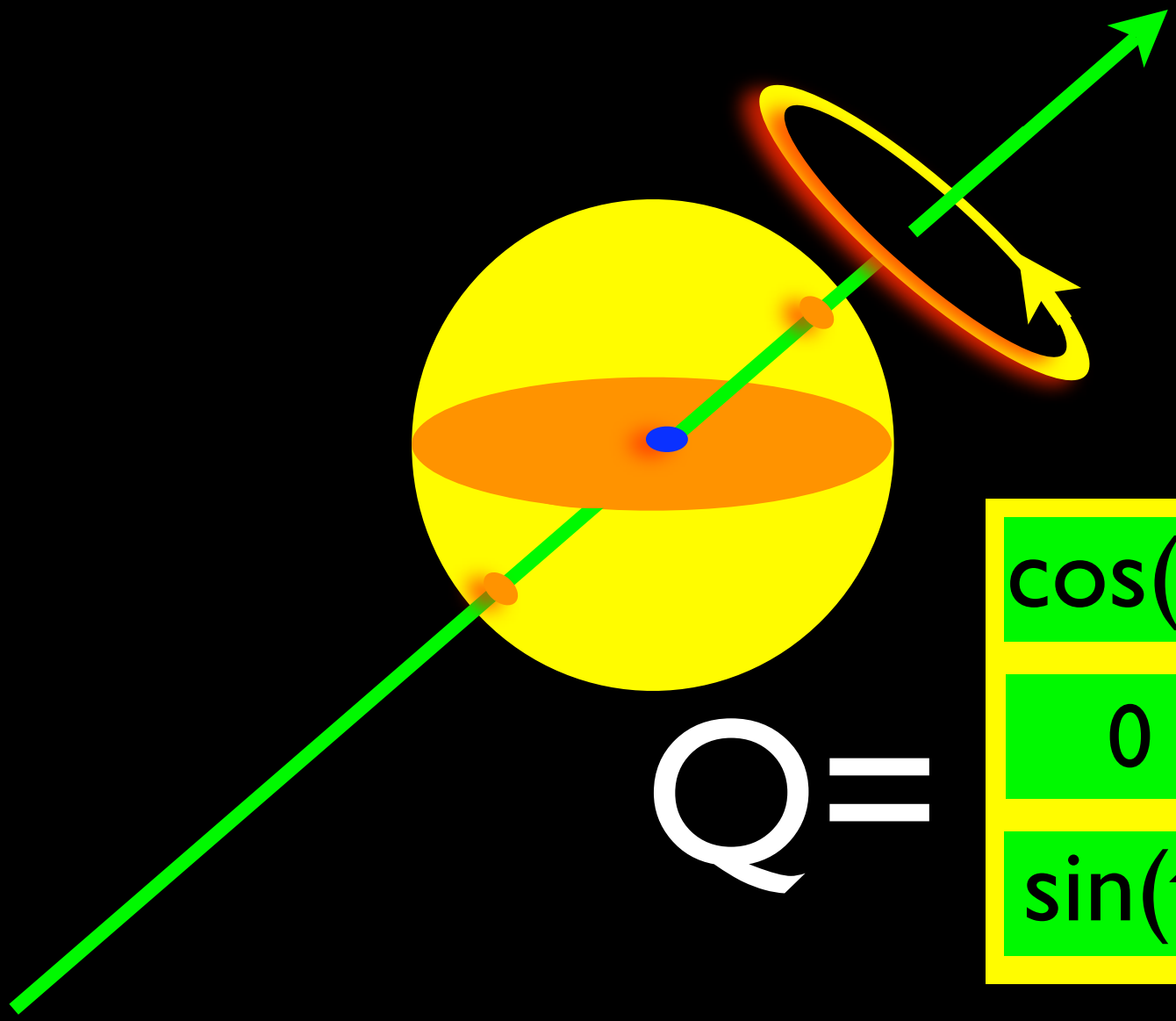
where A is an orthogonal matrix.

$$A^T A = I$$

- typically, reflections or rotations
- preserve length
- preserve angles
- column vectors of Q form orthonormal basis
- determinant of A is either 1 or -1

Rotations

$\det(Q)=1$



$Q=$

example:

cos(t)	0	-sin(t)
0	1	0
sin(t)	0	cos(t)

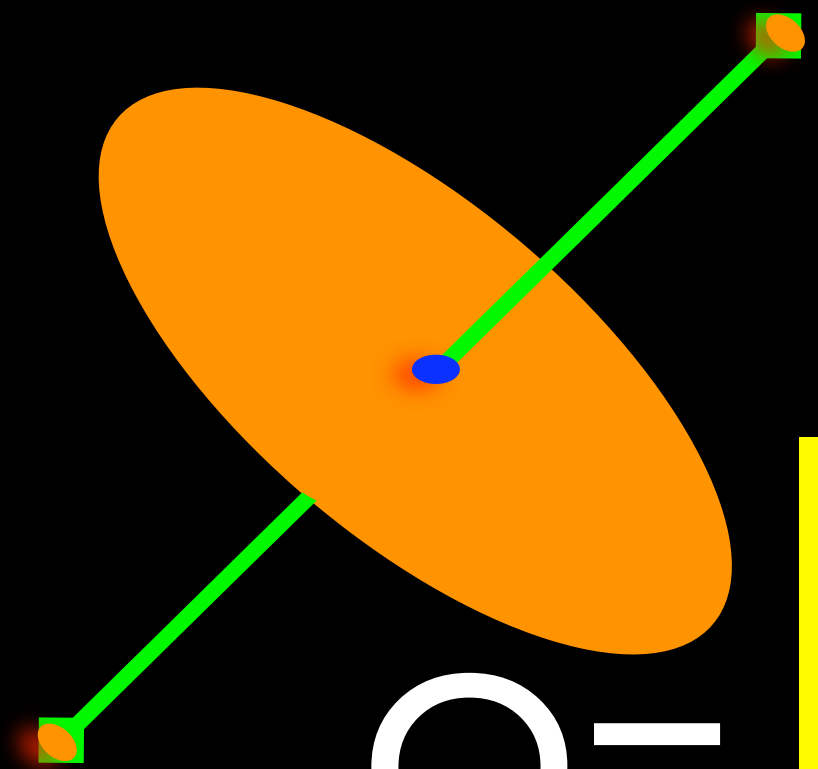
Reflections

$\det(Q)=-1$

if reflected at odd dimensional space

$\det(Q)=+1$

if reflected at even dimensional space



example:

$$Q=\begin{bmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ \sin(t) & 0 & -\cos(t) \end{bmatrix}$$

don't confuse orthogonal transformations
with orthogonal projections
which satisfy for n x n matrices

$$A^2 = A$$

in general
not invertible

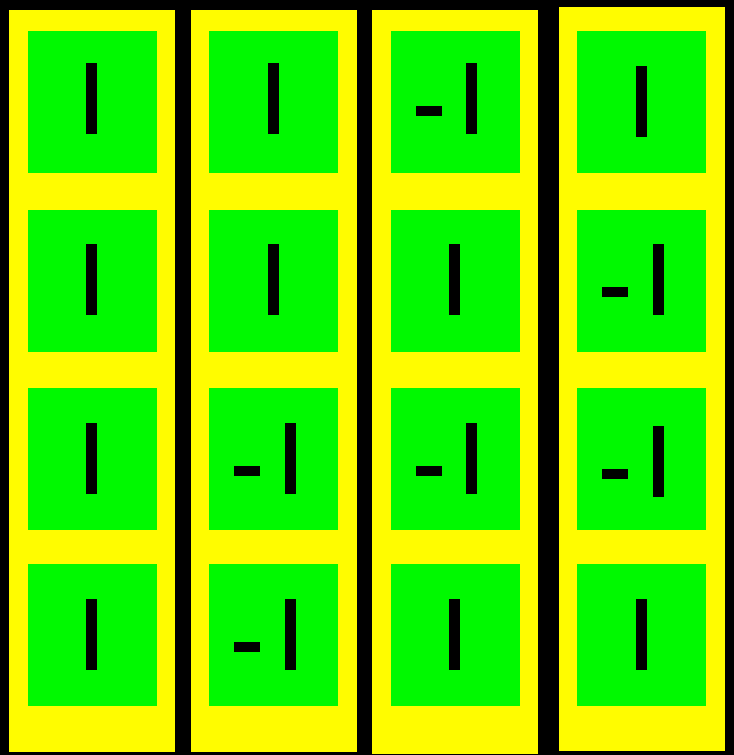
Example 1:

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

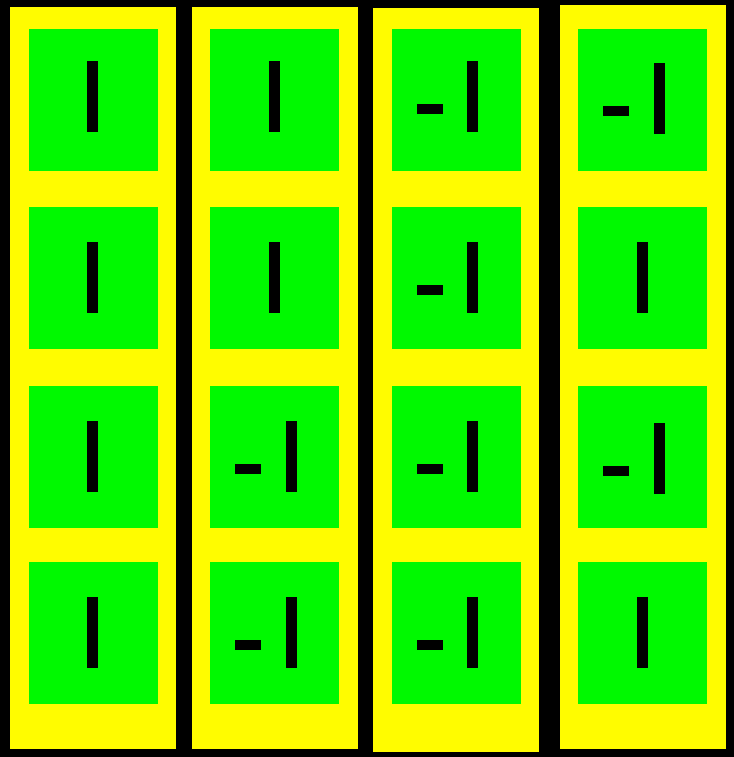
Example 2:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which of the two matrices is an orthogonal matrix?



/2



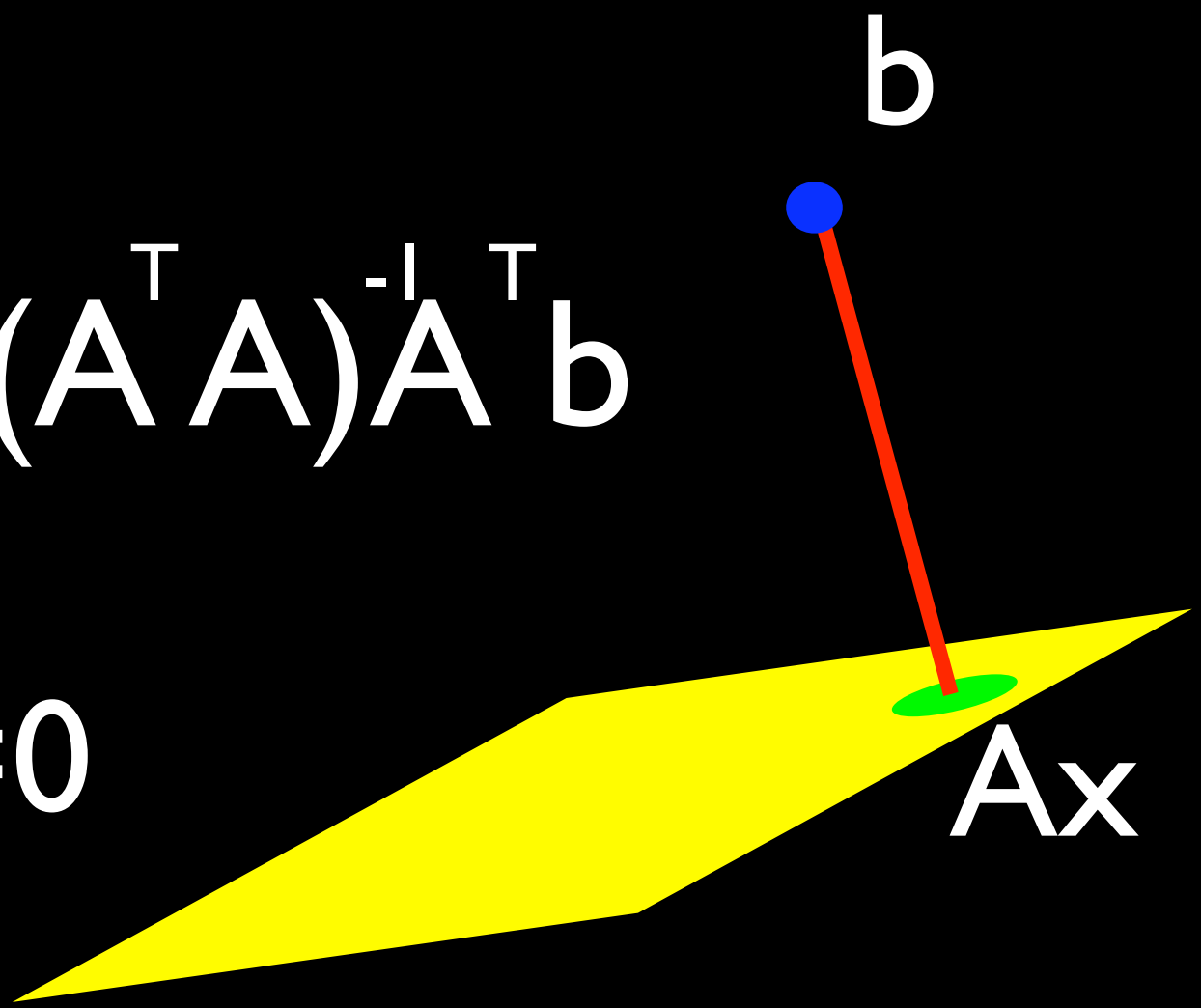
/2

Least Square Solution

solve $Ax=b$ as good as possible

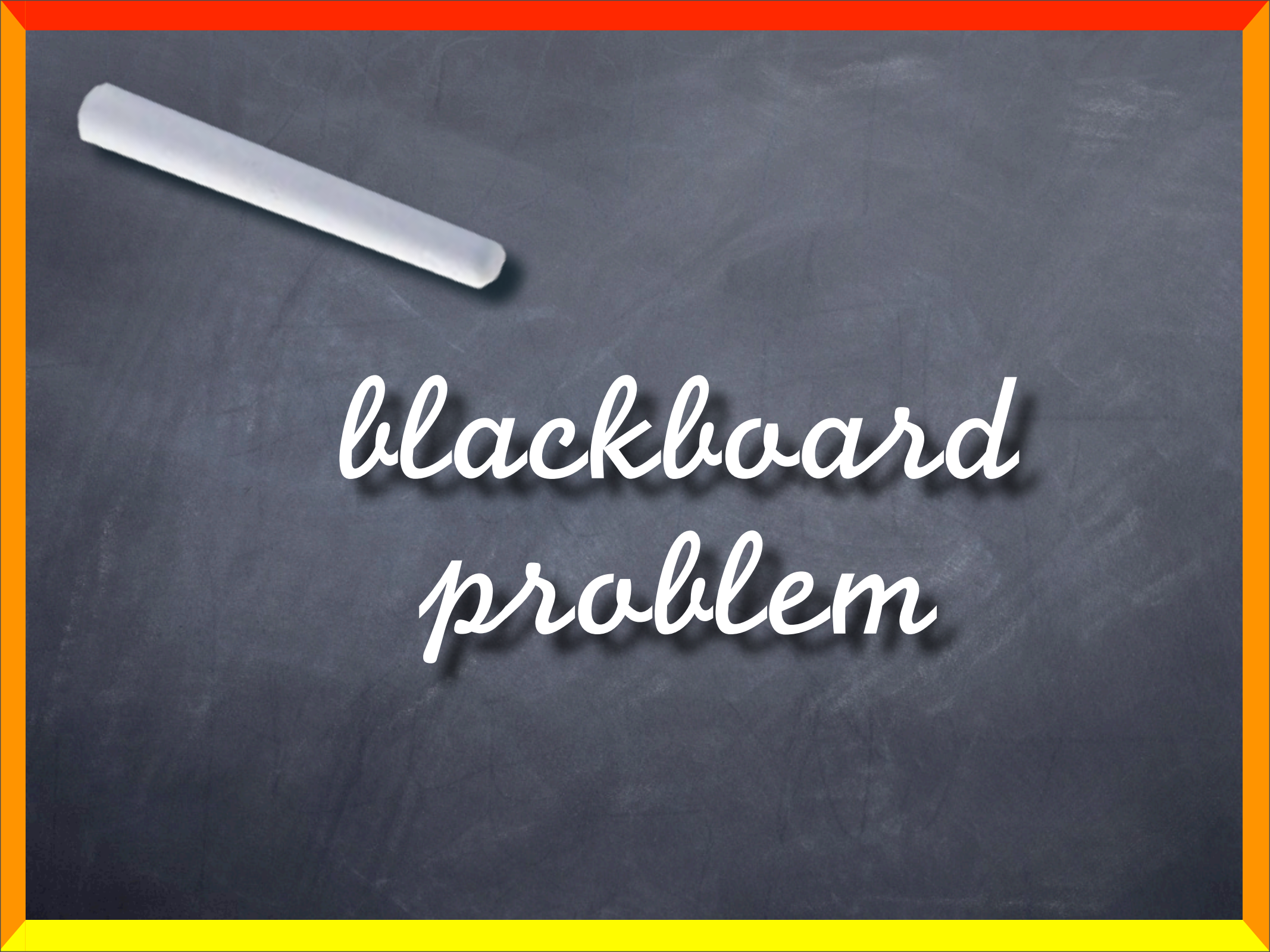
$$x_* = (A^T A)^{-1} A^T b$$

$$A^T(Ax-b)=0$$



Projection

$$P\,y = A(A^{T}A)^{-1}A^{T}\,y$$


blackboard
problem

Find the projection formula onto the plane spanned by the two vectors

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

answer:

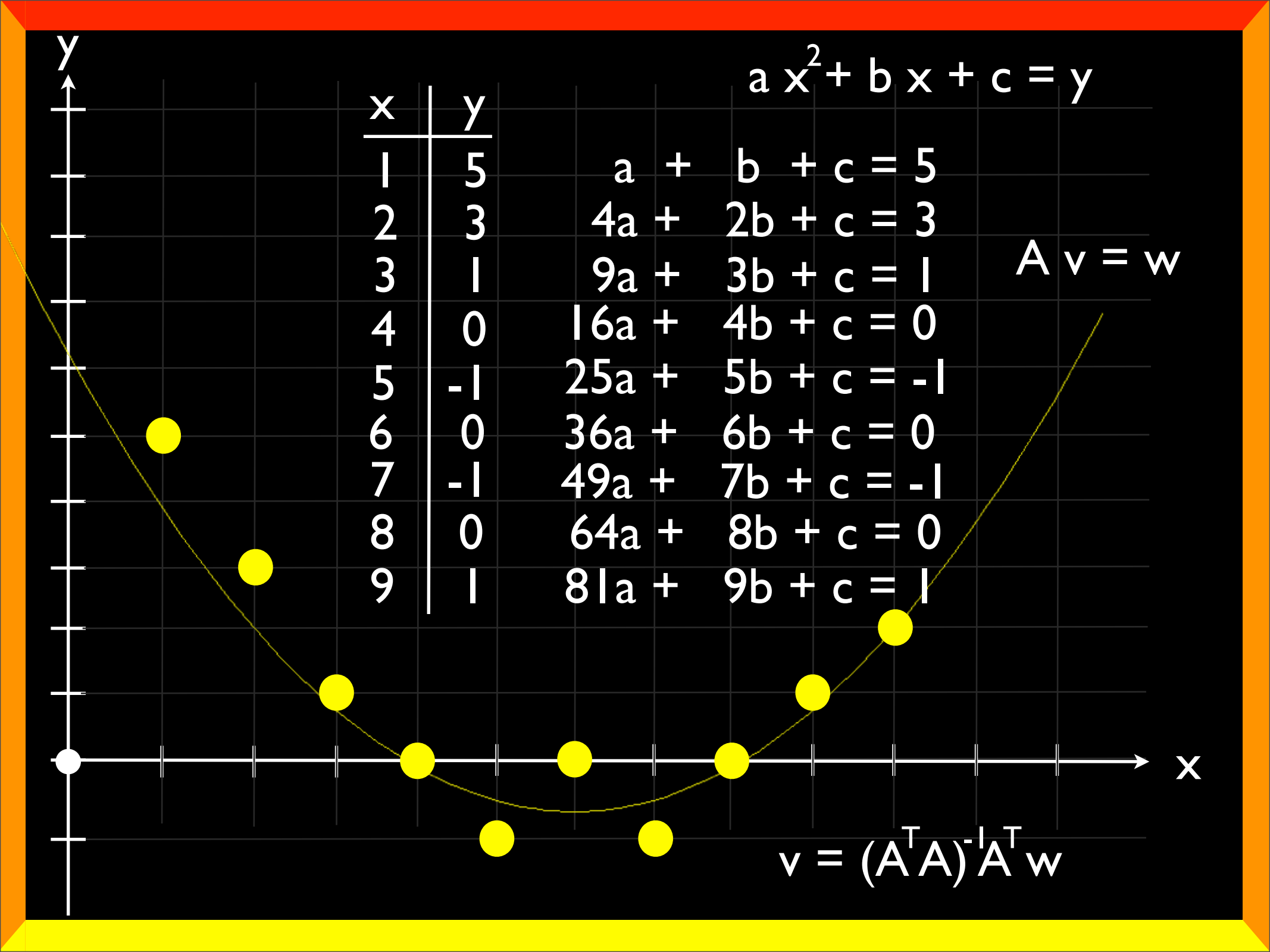
$$P = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 2 & 0 & 4 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 \end{bmatrix} /6$$

Data fitting

To fit data with a set of functions, just plug in the data

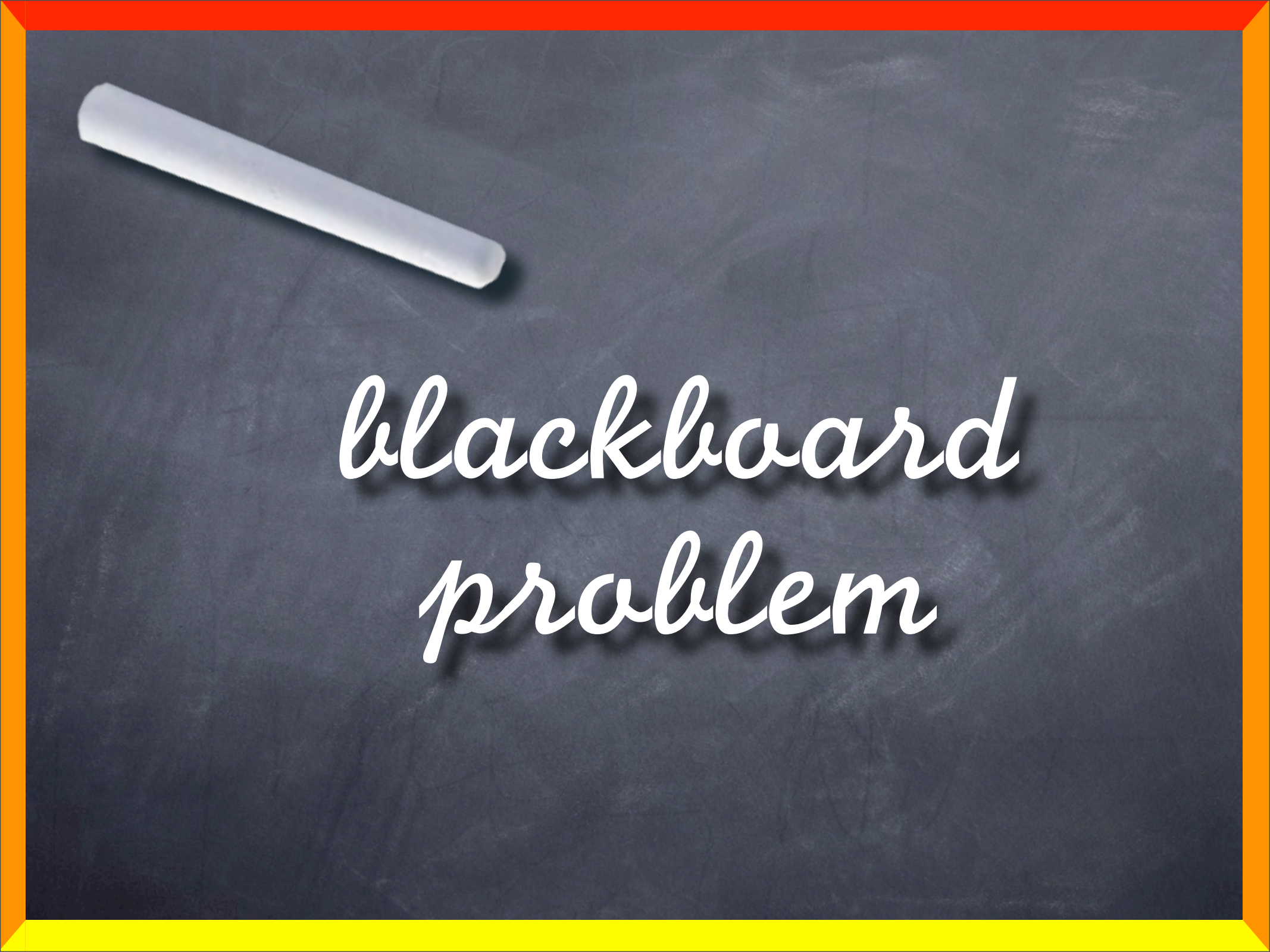
$$y_i = f_k(x)_i .$$

Here is an example with functions $1, x, x^2$



x	y
1	5
2	3
3	1
4	0
5	-1
6	0
7	-1
8	0
9	1

$$a + b + c = 5$$
$$4a + 2b + c = 3$$
$$9a + 3b + c = 1$$
$$16a + 4b + c = 0$$
$$25a + 5b + c = -1$$
$$36a + 6b + c = 0$$
$$49a + 7b + c = -1$$
$$64a + 8b + c = 0$$
$$81a + 9b + c = 1$$

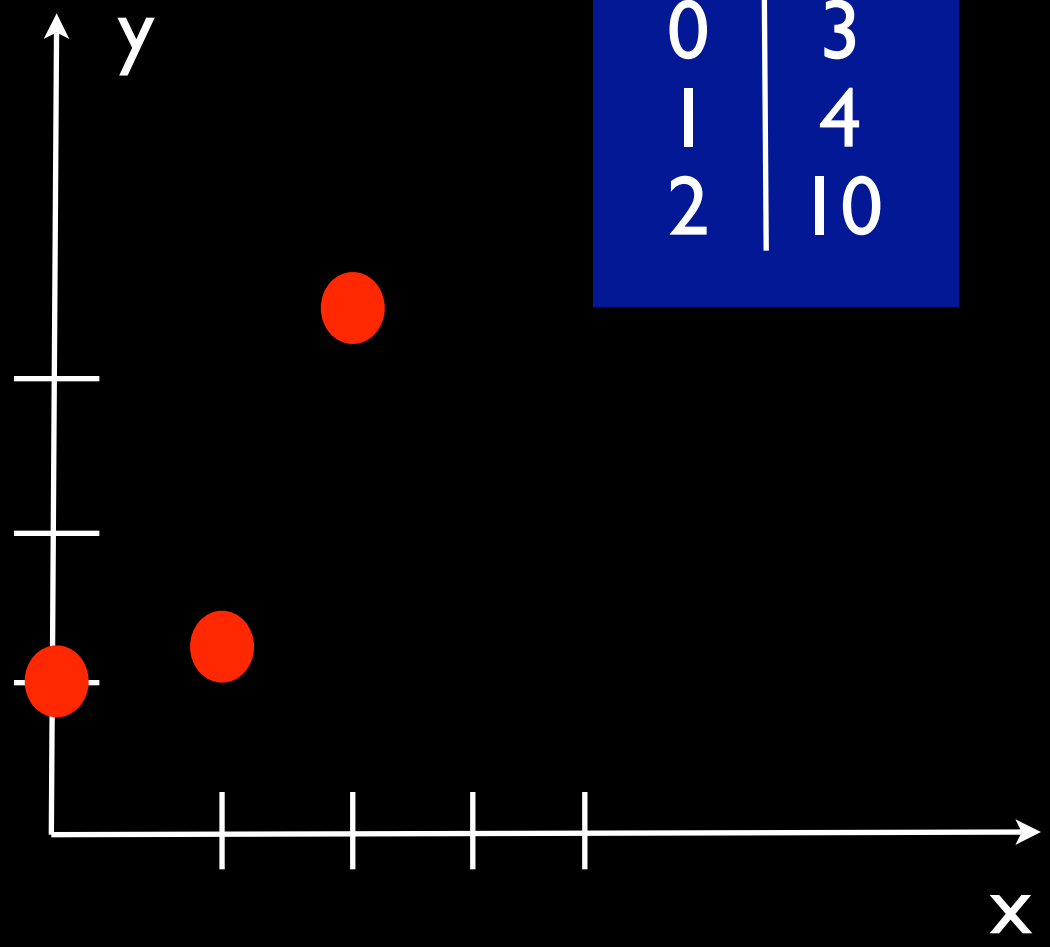

blackboard
problem

Fit the following data with functions

$f_1(x) = \cos(\pi x/2)$

$f_2(x) = x + 1$

x	y
0	3
1	4
2	10



Solution:

x	y
0	3
1	4
2	10

$a\cos(\pi x/2) + b(x+1) = y$

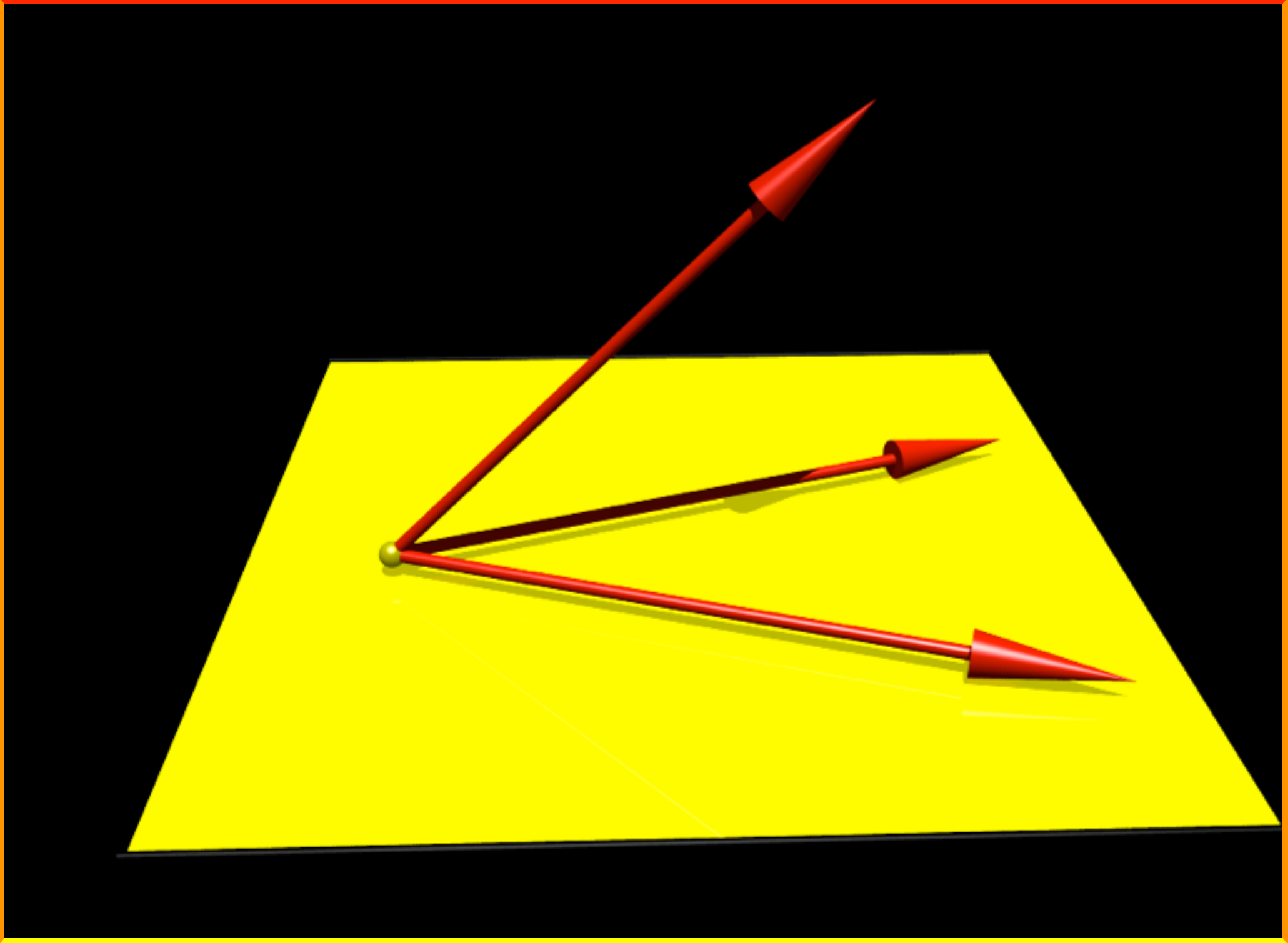
$a\,1 + b\,1 = 3$
$a\,0 + b\,2 = 4$
$a\,(-1) + b\,3 = 10$

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} \qquad A^T b = \begin{bmatrix} -7 \\ 41 \end{bmatrix}$$

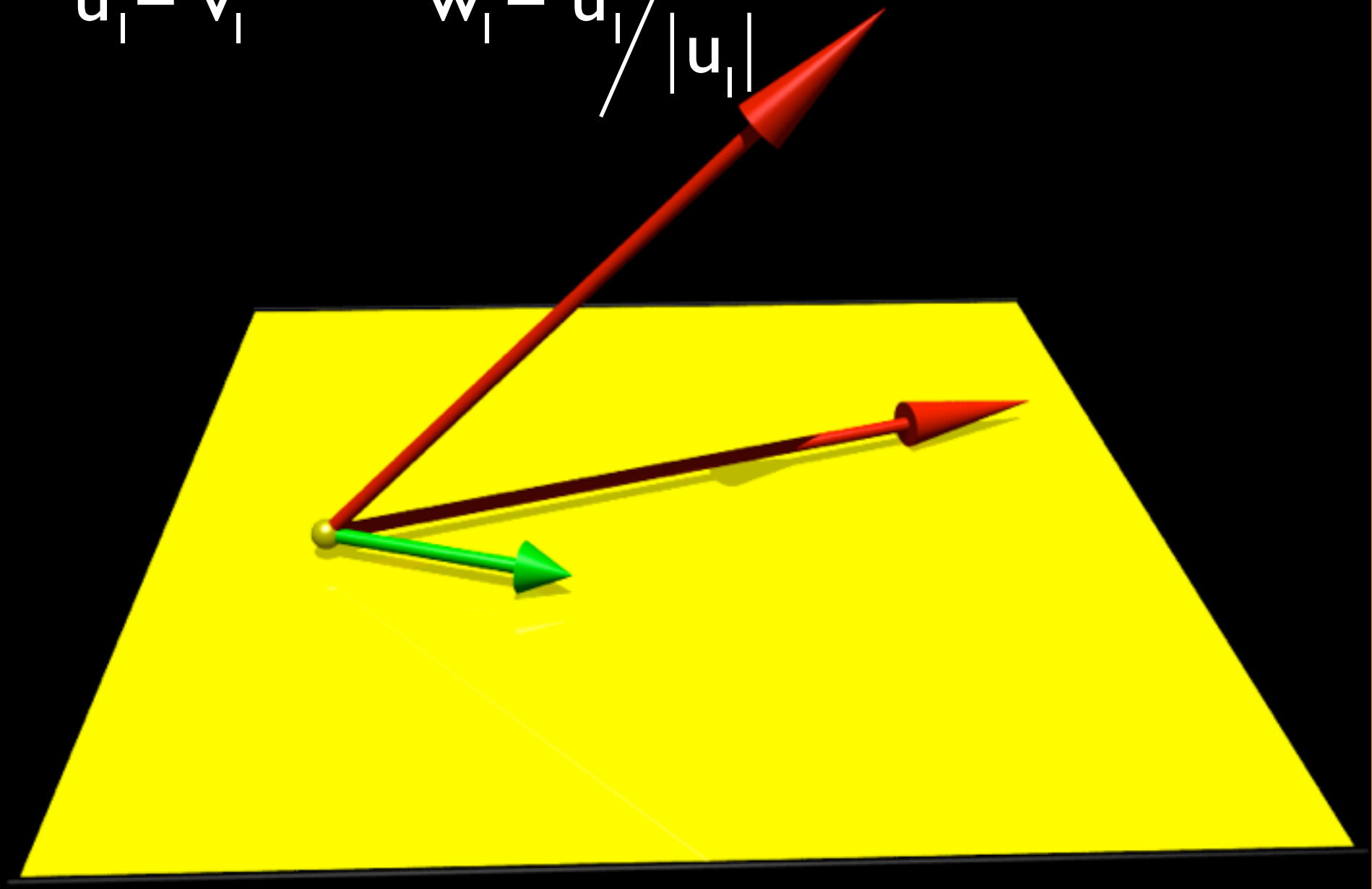
$$A^T A = \begin{bmatrix} 2 & -2 \\ -2 & 14 \end{bmatrix} \qquad (A^T A)^{-1} = \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix} /12$$

$$(A^T A)^{-1} A^T b = \begin{bmatrix} -2/3 \\ 17/6 \end{bmatrix}$$

Gram Schmidt
QR Decomposition

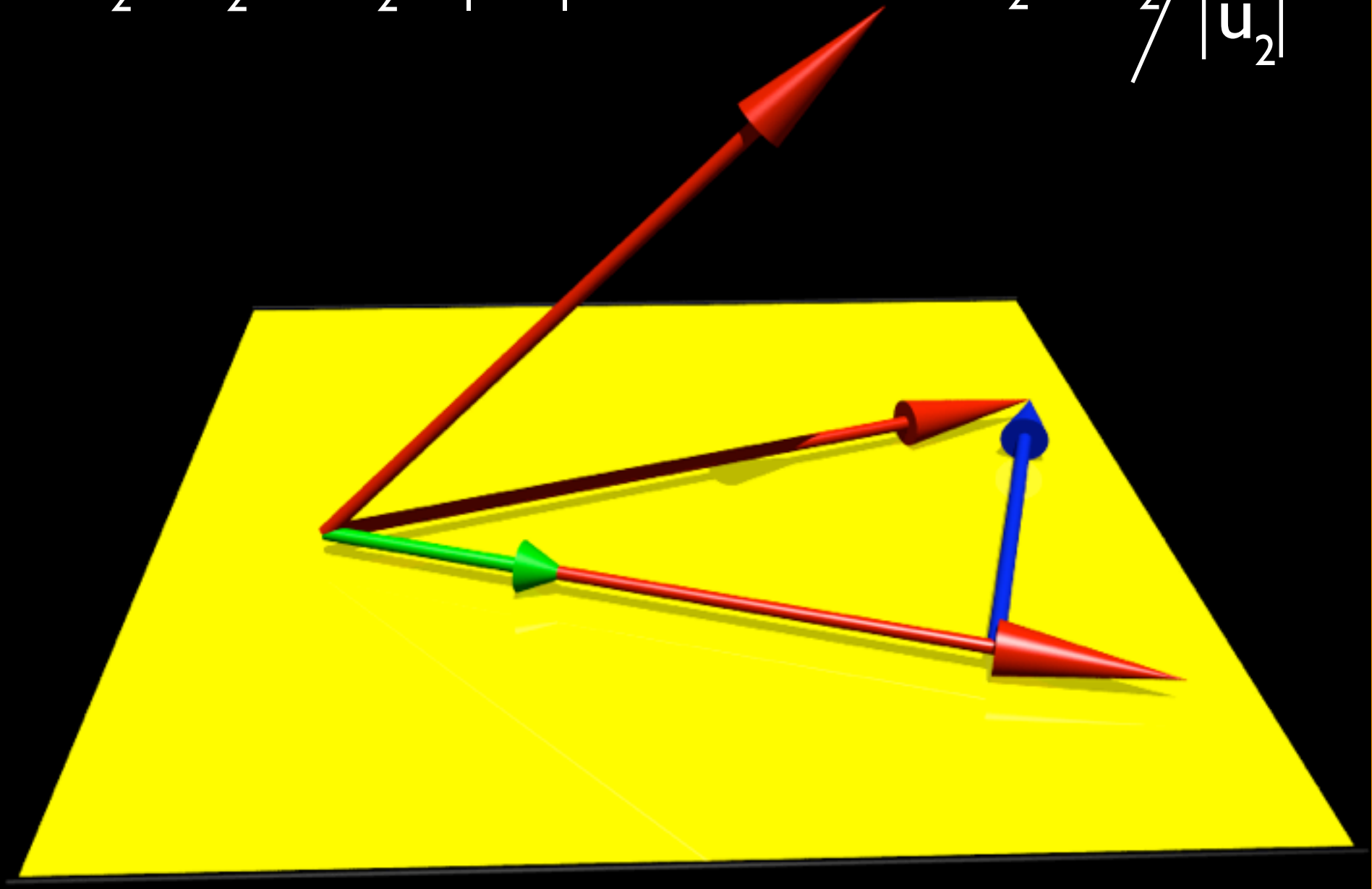


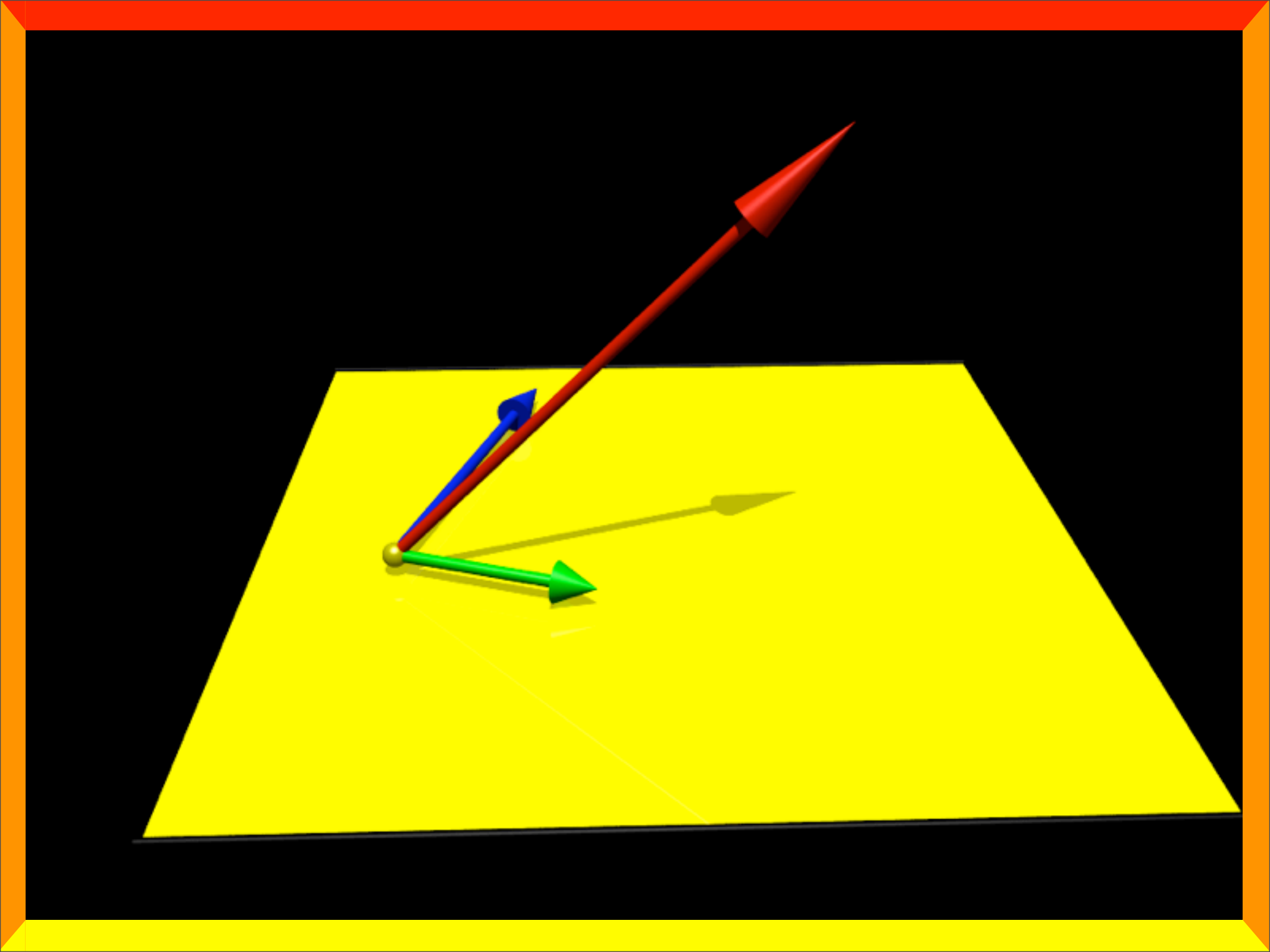
$u_1 = v_1$ $w_1 = u_1 / |u_1|$

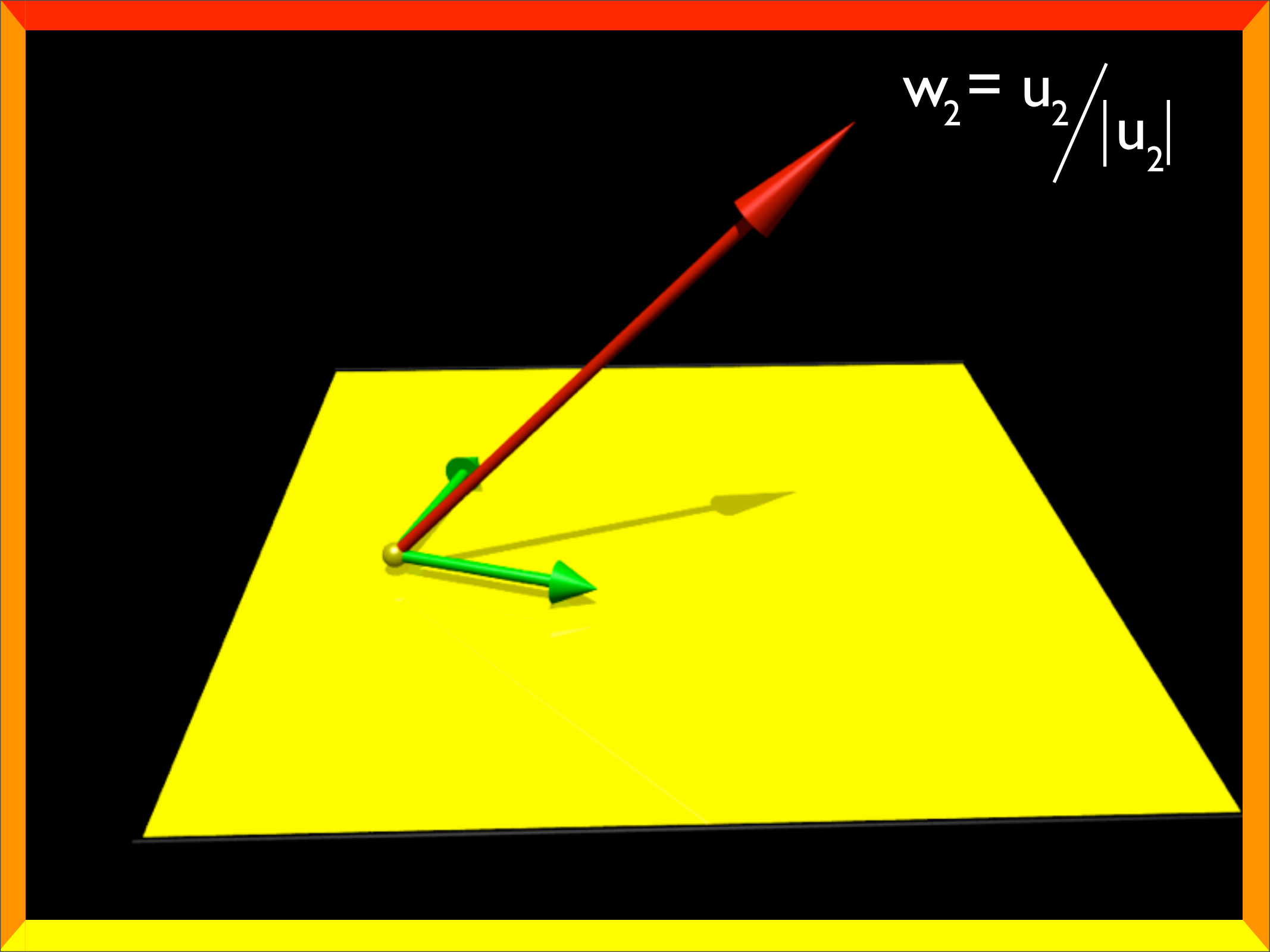


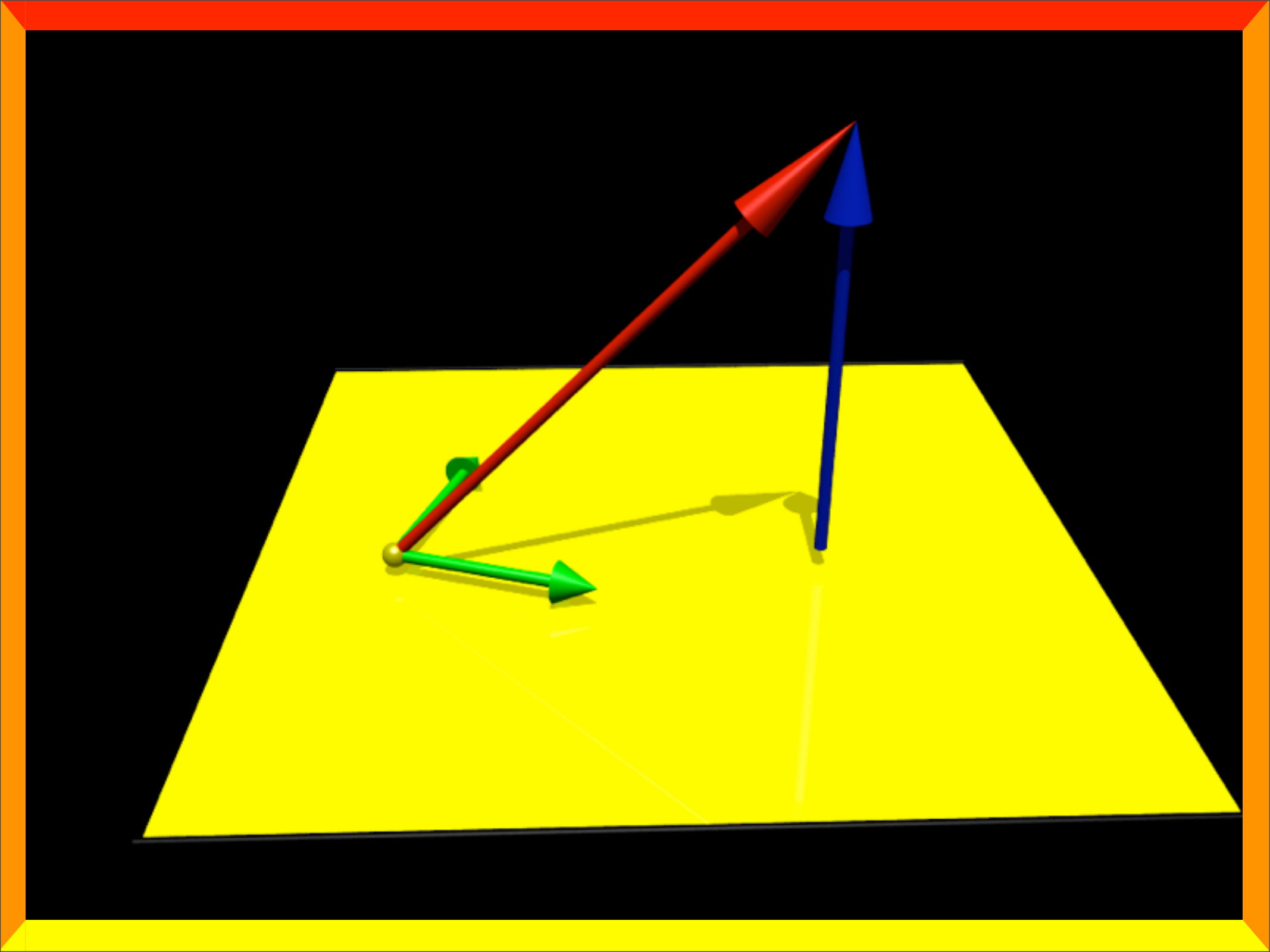
$u_2 = v_2 - (v_2 \cdot w_1)\, w_1$

$w_2 = u_2 / |u_2|$



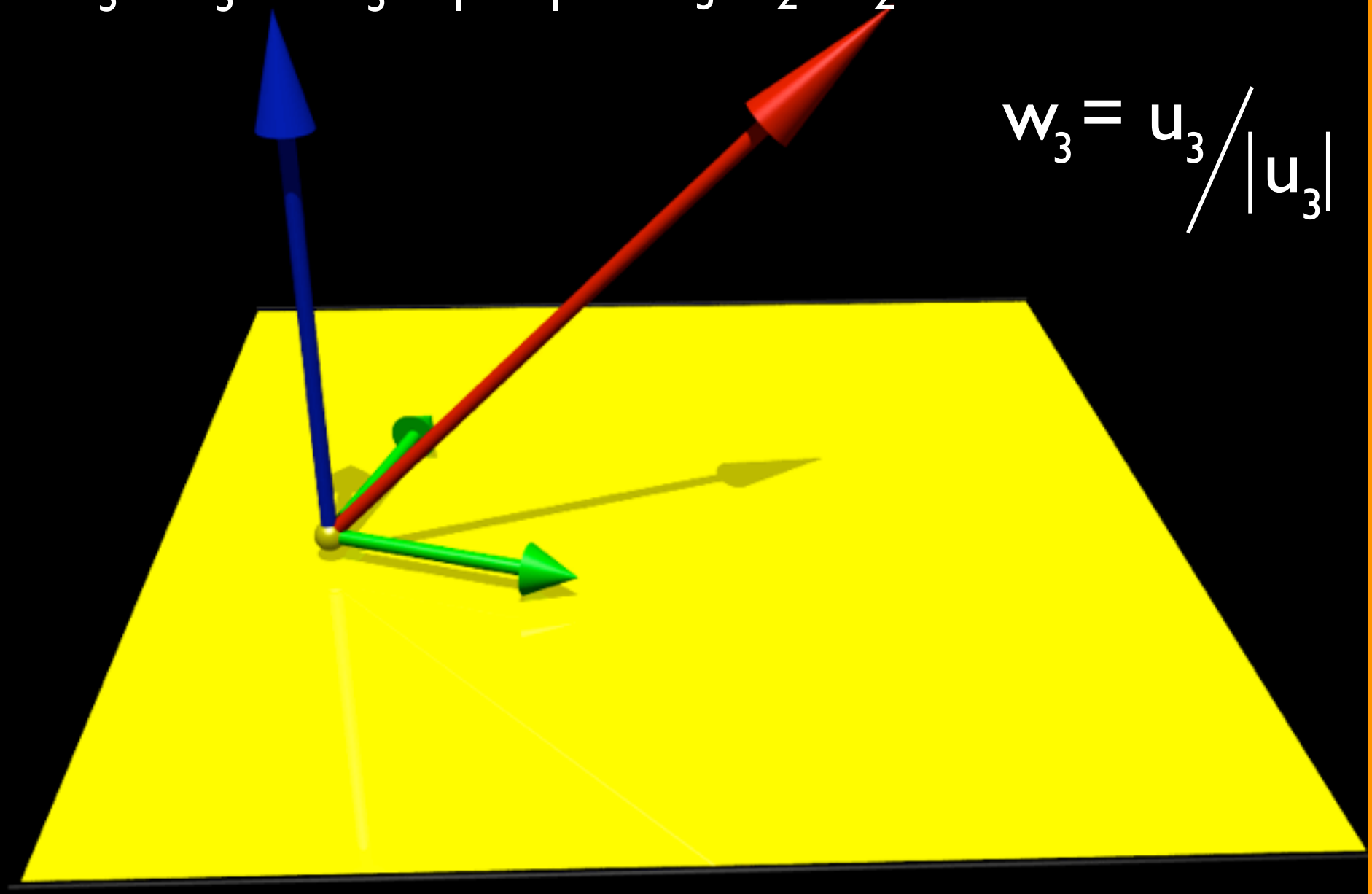


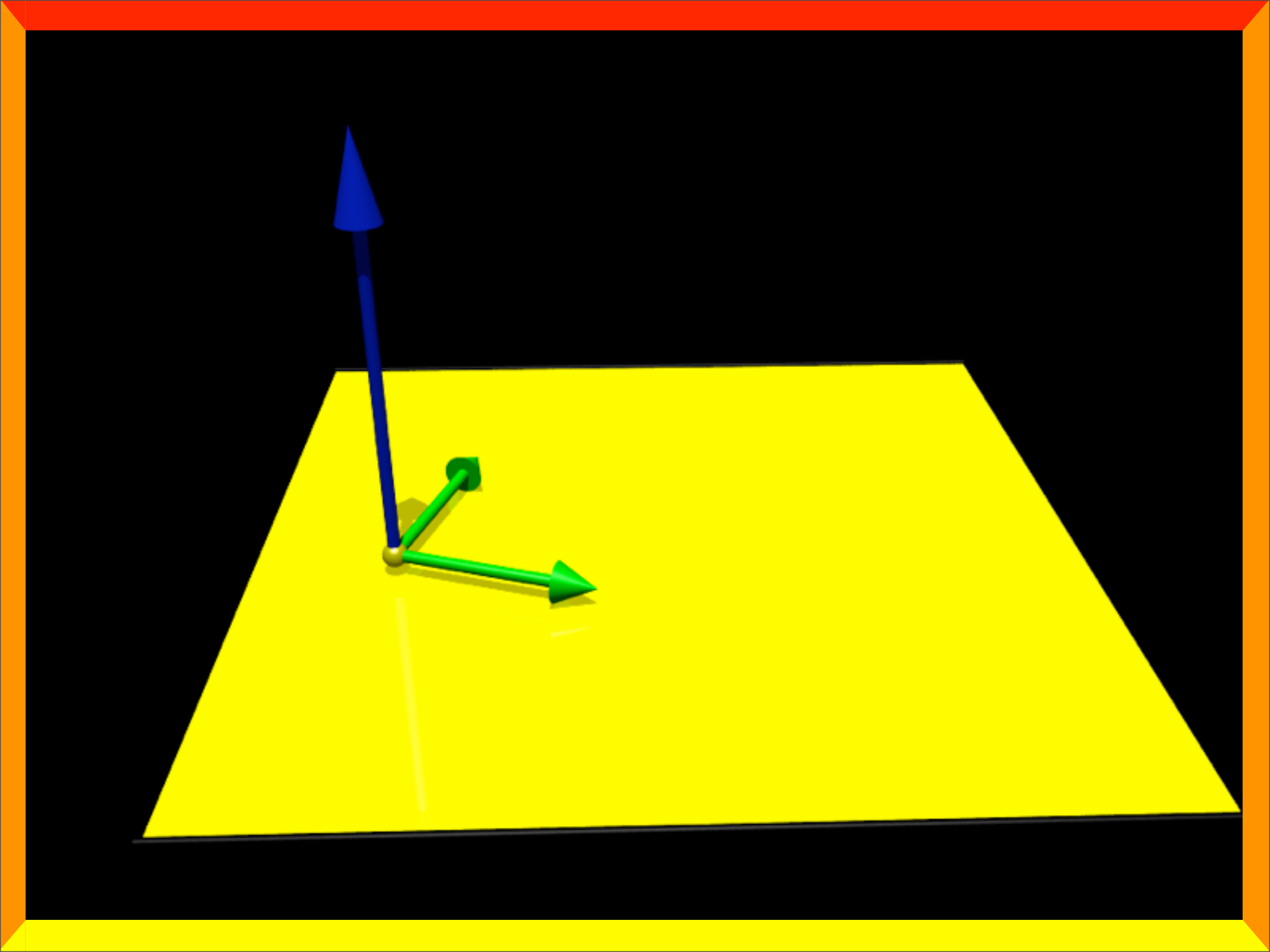

$w_2 = u_2 / |u_2|$

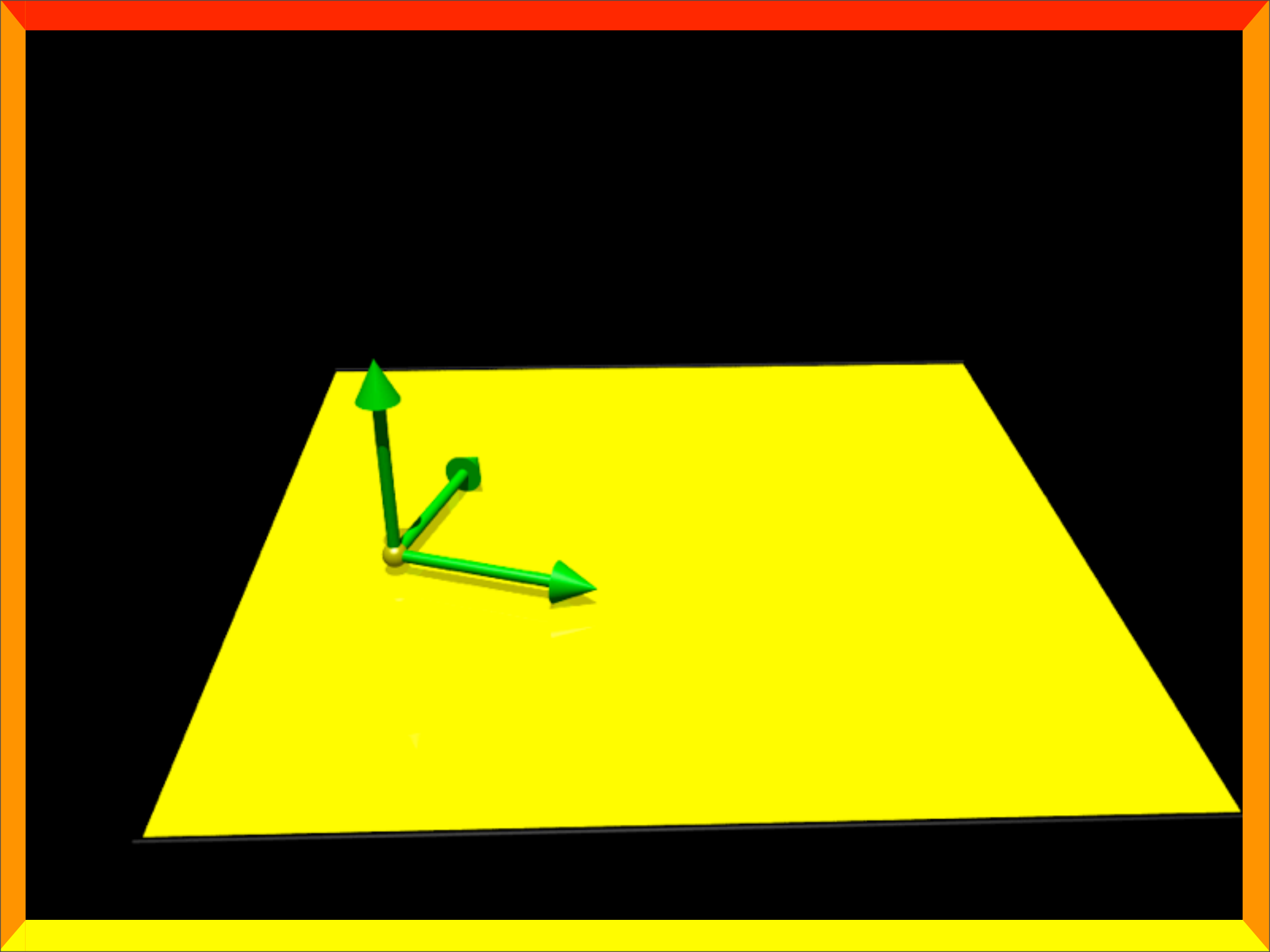


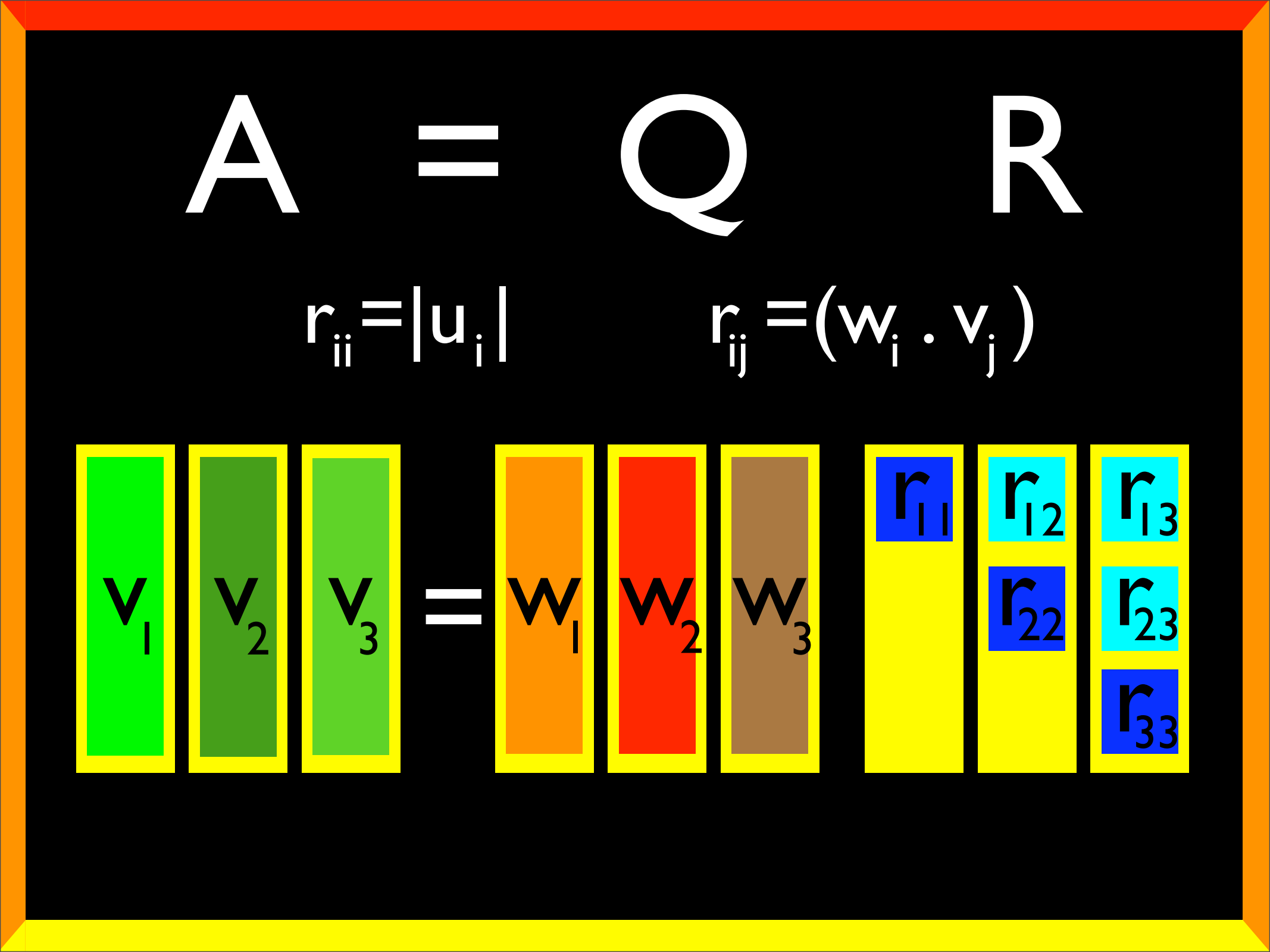
$$u_3 = v_3 - (v_3 \cdot w_1)\, w_1 - (v_3 \cdot w_2)\, w_2$$

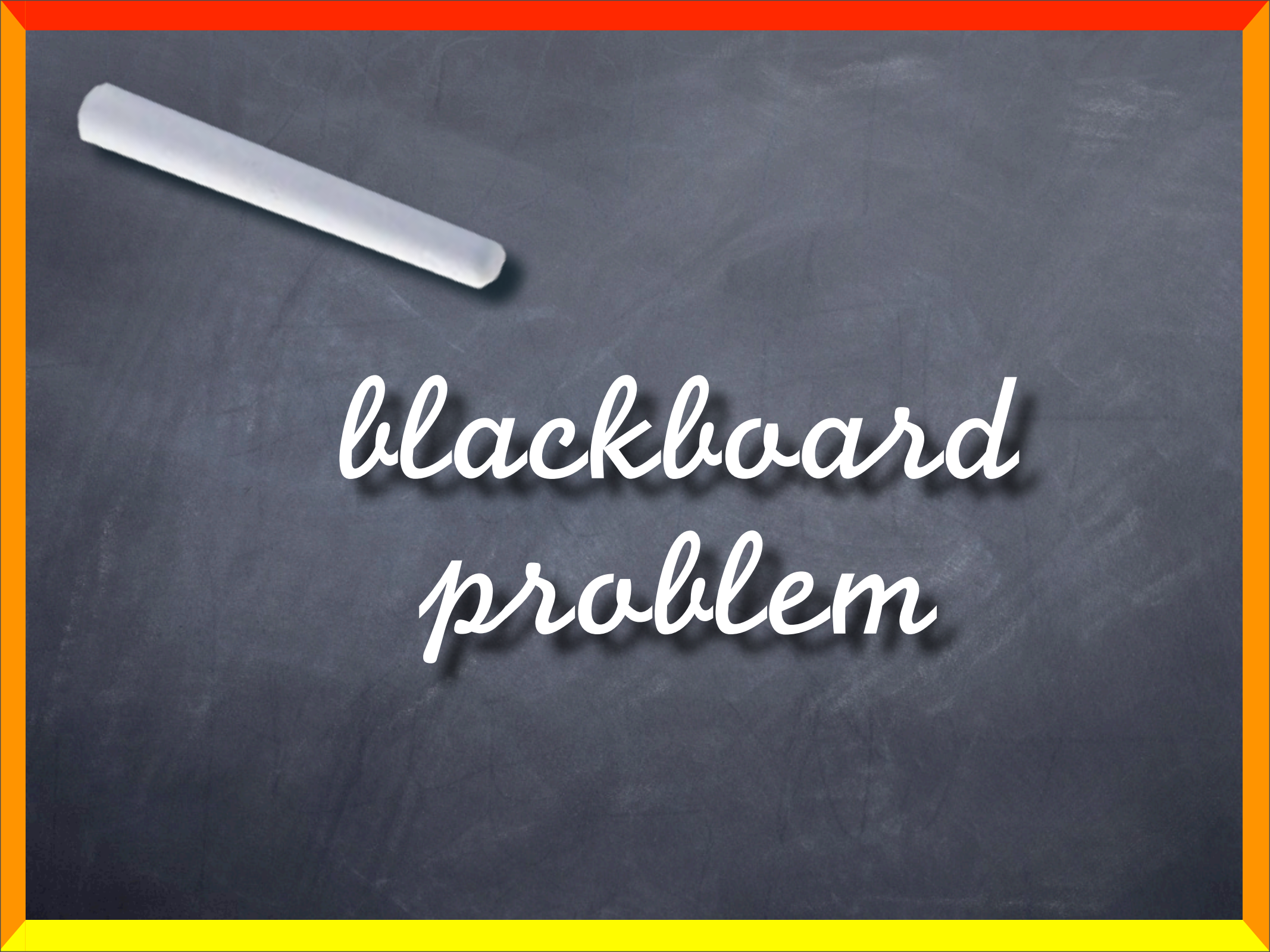
$$w_3 = u_3 / |u_3|$$

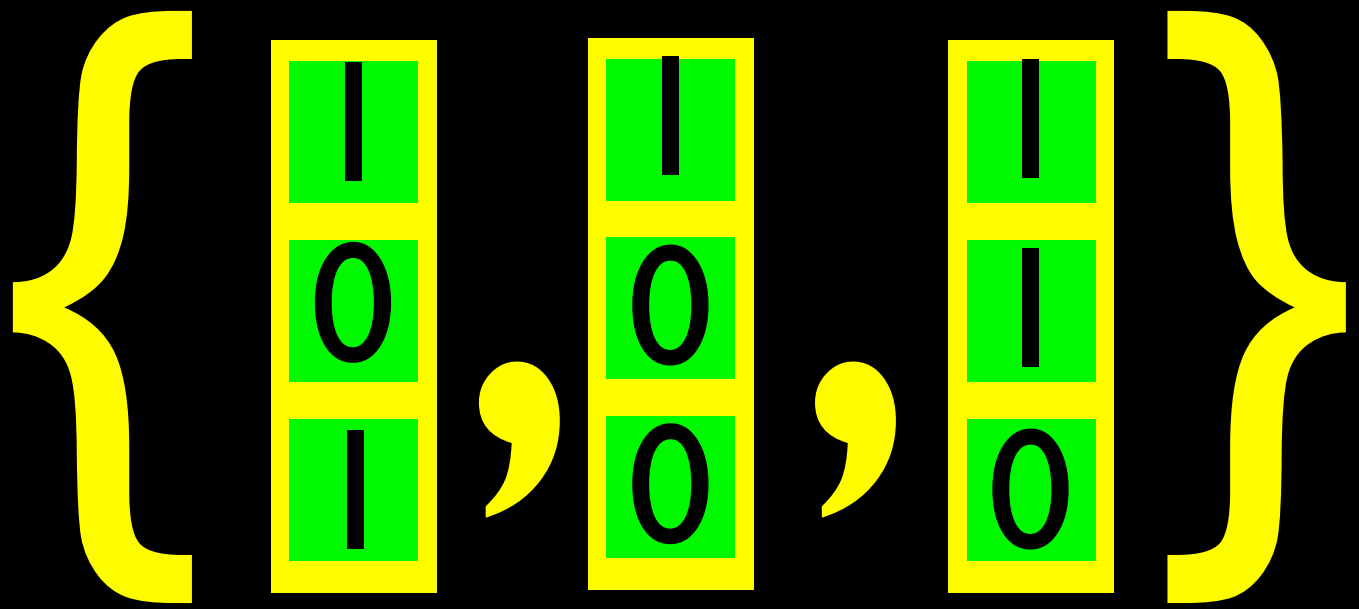







A = Q R
$r_{ii} = |u_i|$
$r_{ij} = (w_i \cdot v_j)$
v_1 v_2 v_3 = w_1 w_2 w_3
r_{11} r_{12} r_{13}
r_{22} r_{23}
r_{33}


blackboard
problem



Apply Gram-Schmidt to this basis and do QR decomposition.

Determinants

$$\det(\,a\,)=a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ad-bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

$$+ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

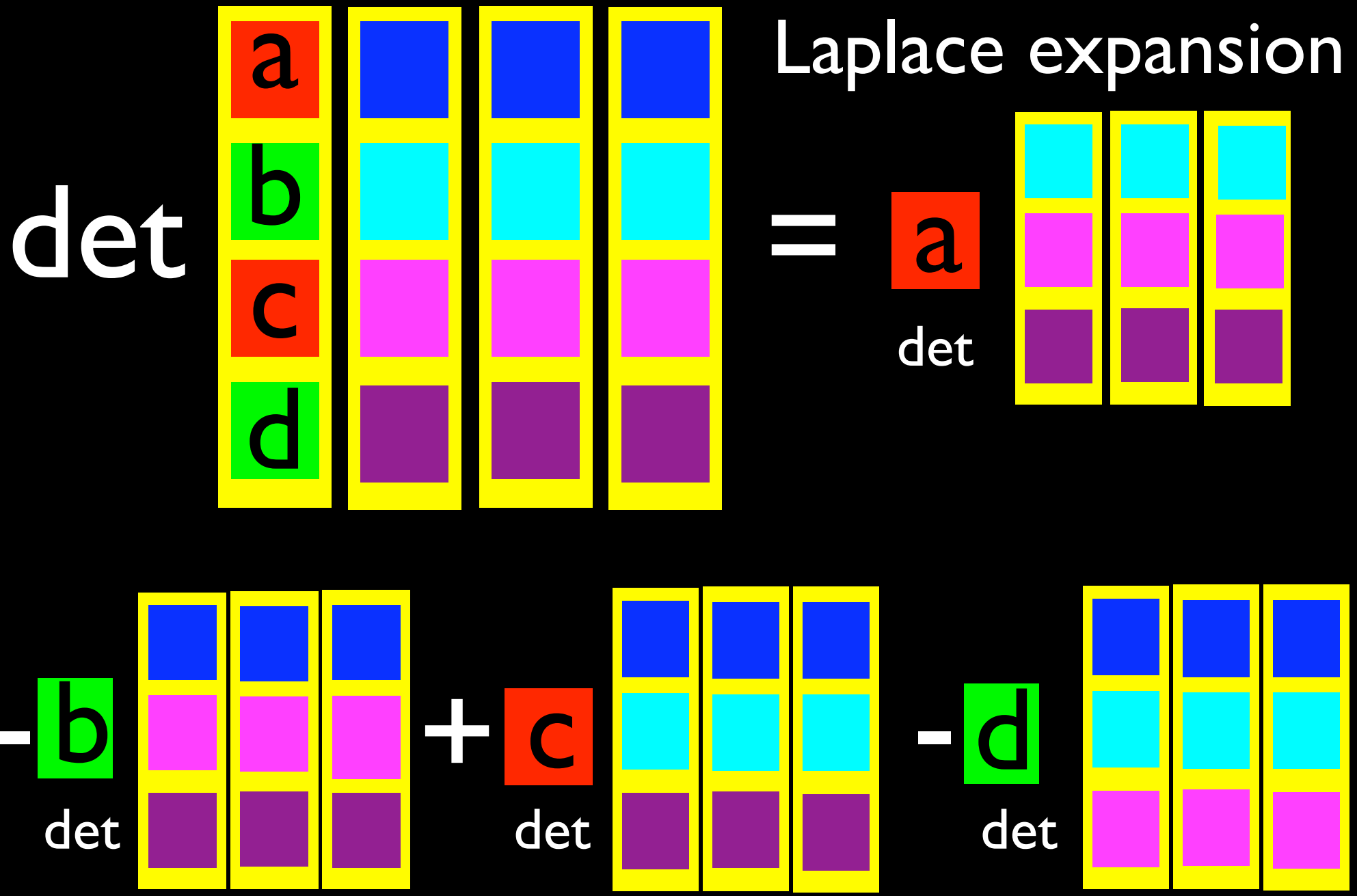
$$aei + bfg + cdh - bdi + ceg + ahf$$

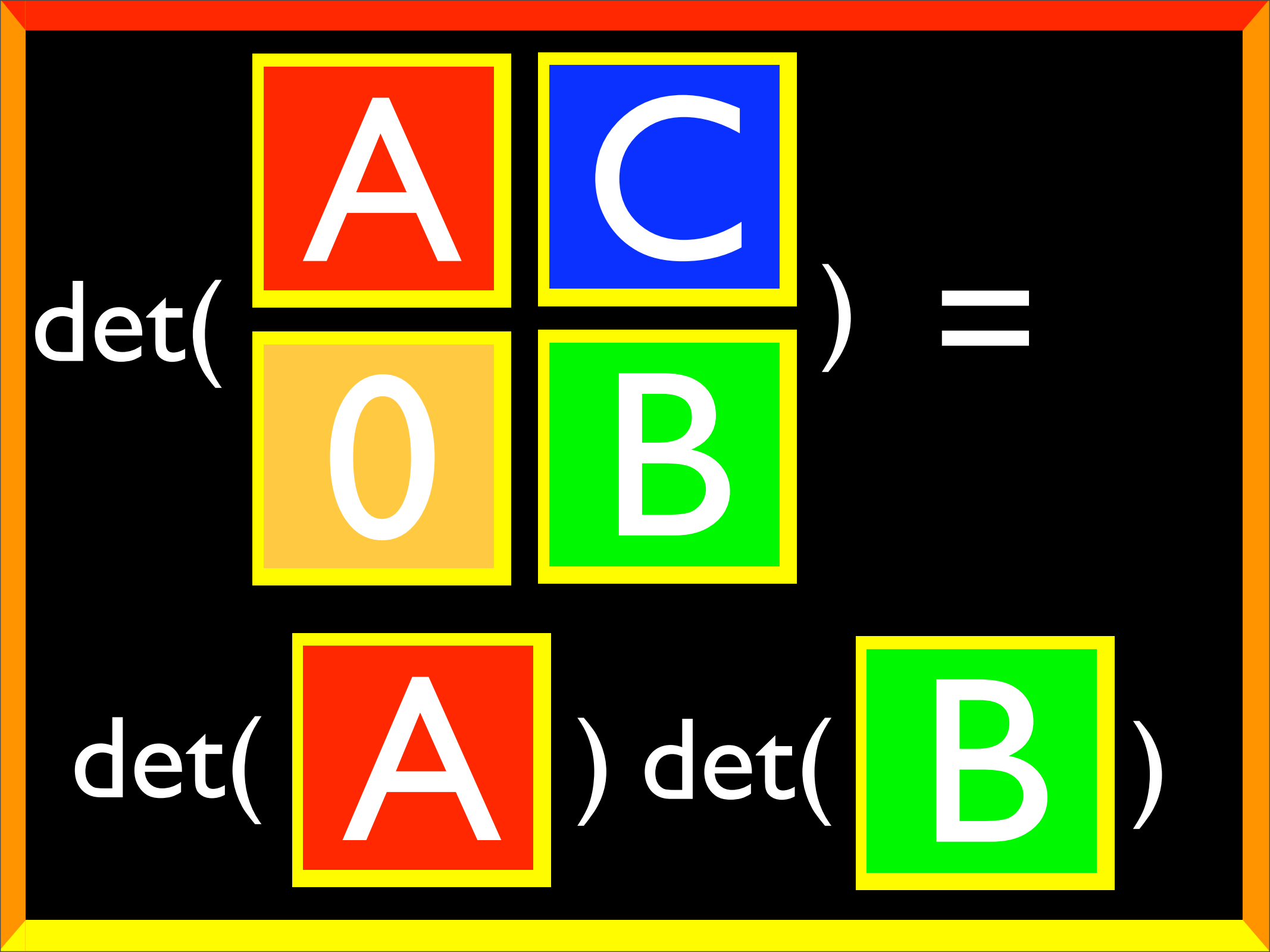
which by the way can be remembered much better with the following memnonic poem:

$$\det\begin{pmatrix} a & e & u \\ n & f & l \\ i & o & k \end{pmatrix} =$$

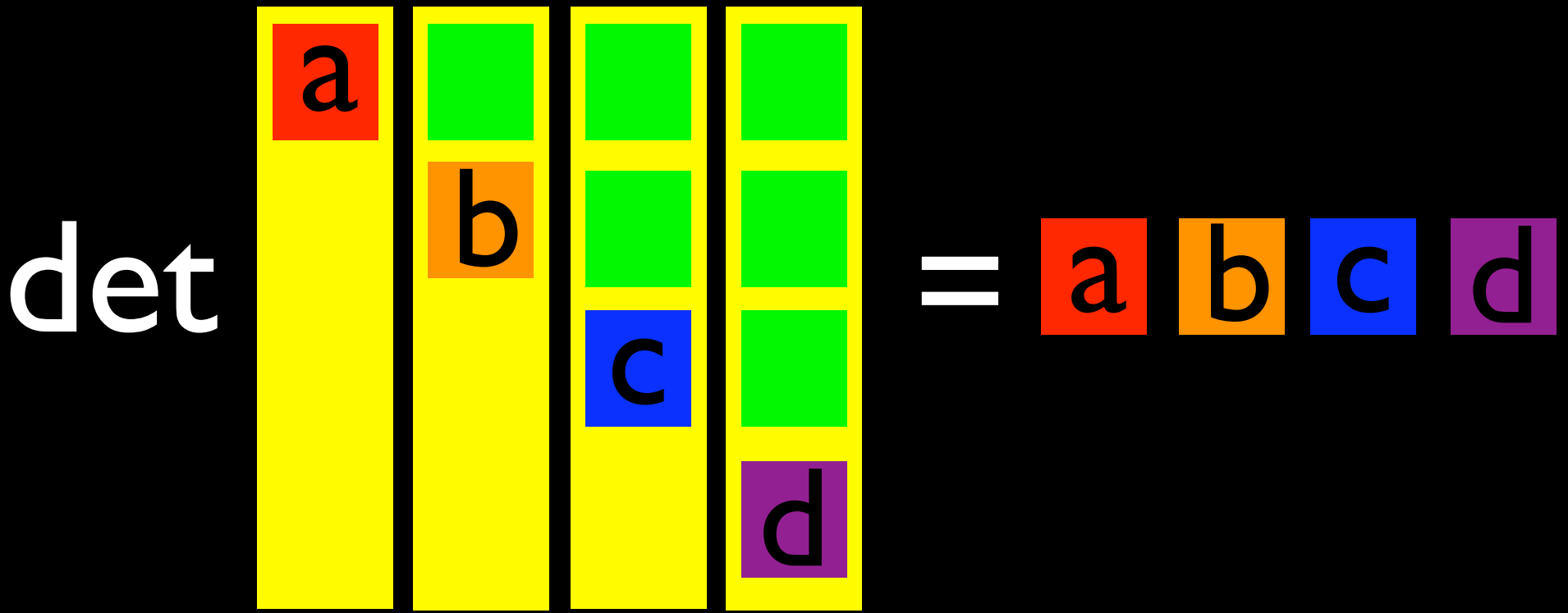
fak+uno+lie - ola-ken-fui

Laplace expansion

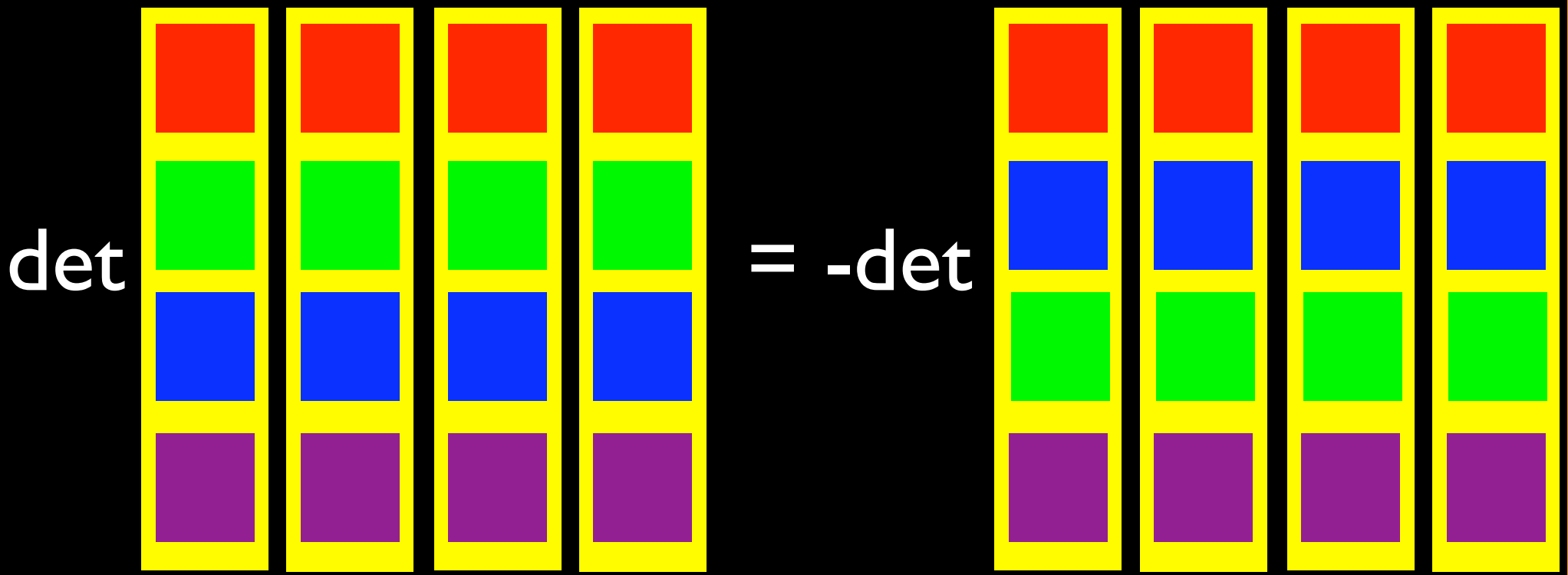



det(A C 0 B) =
det(A) det(B)

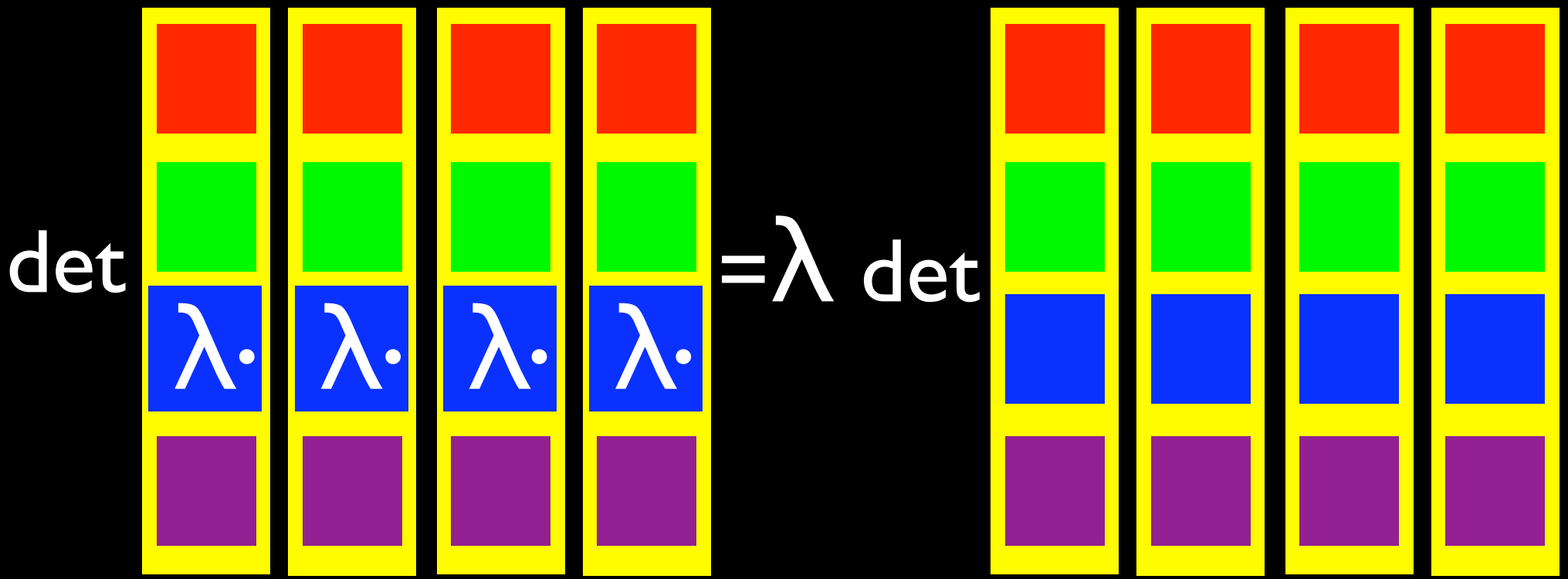
upper or lower triangular matrices



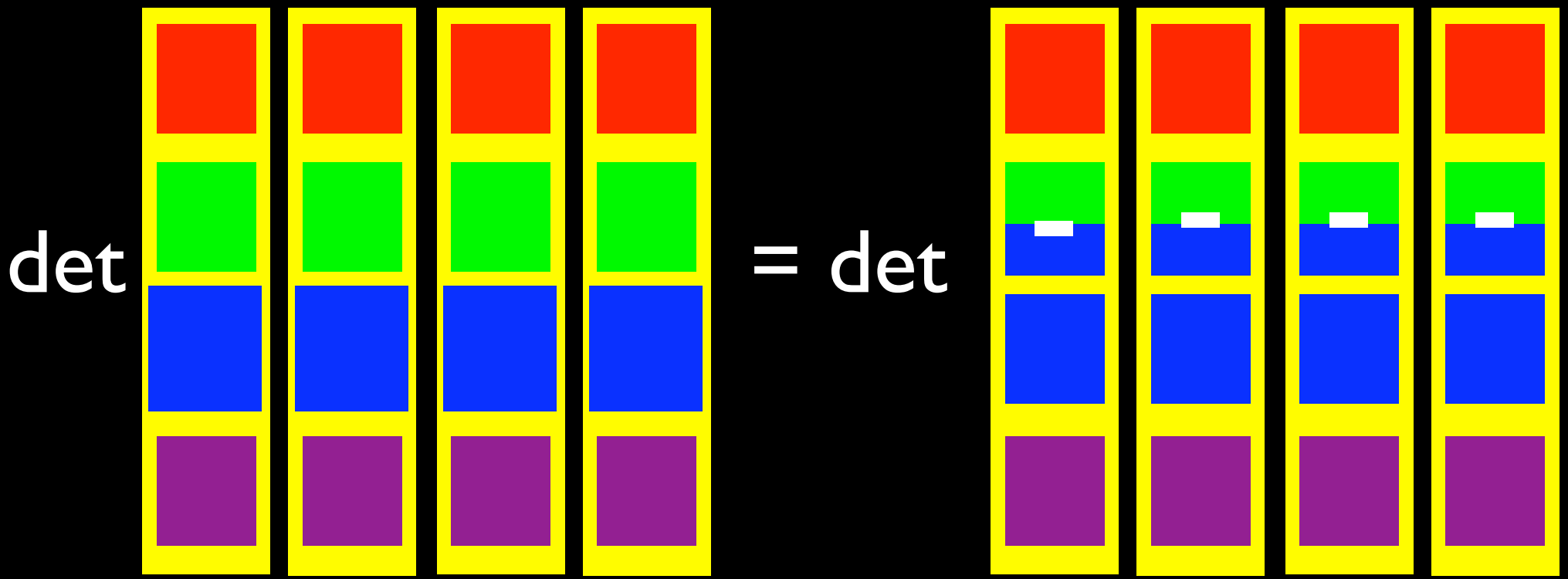
row reduction: swap

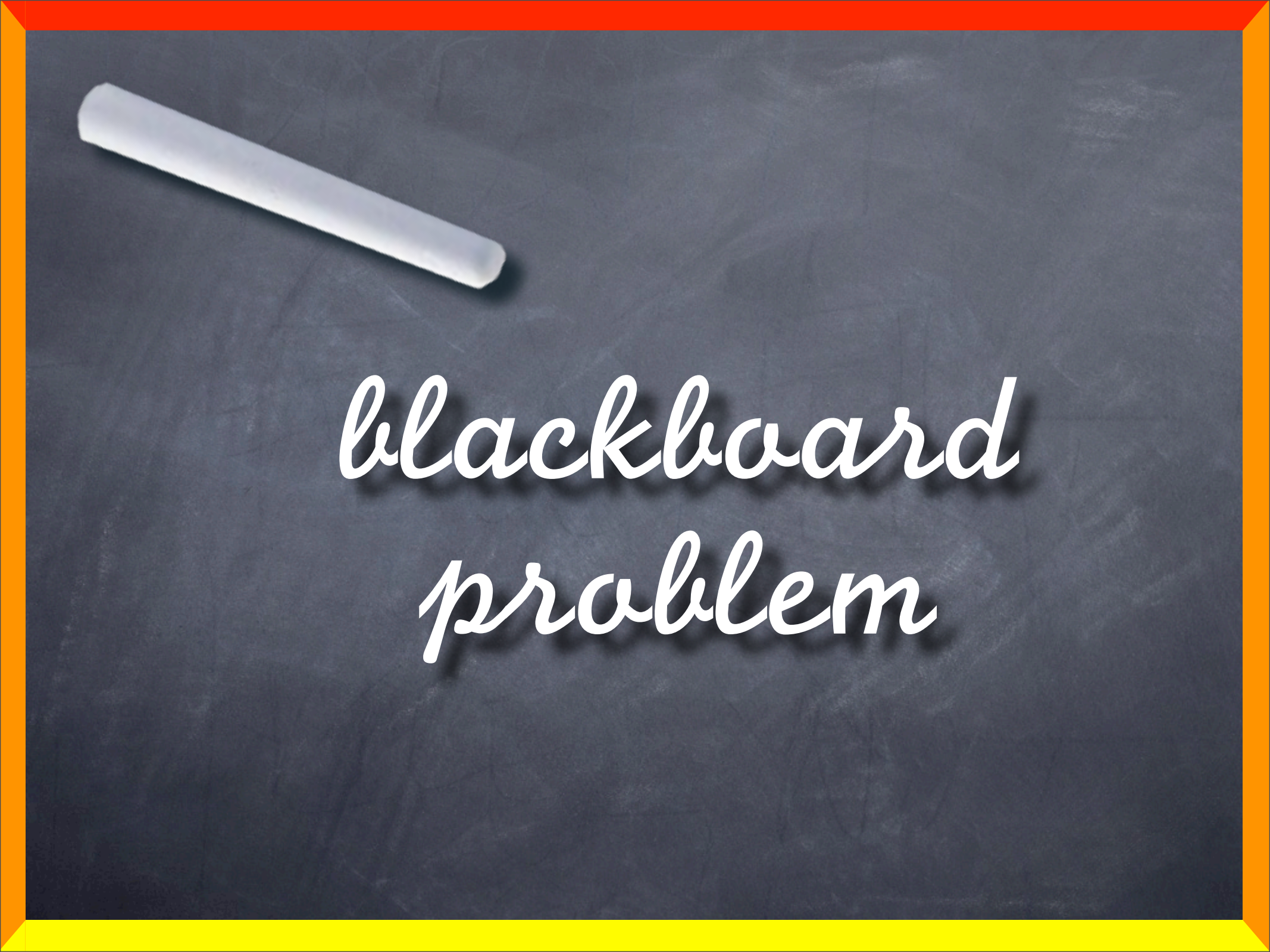


row reduction: scale

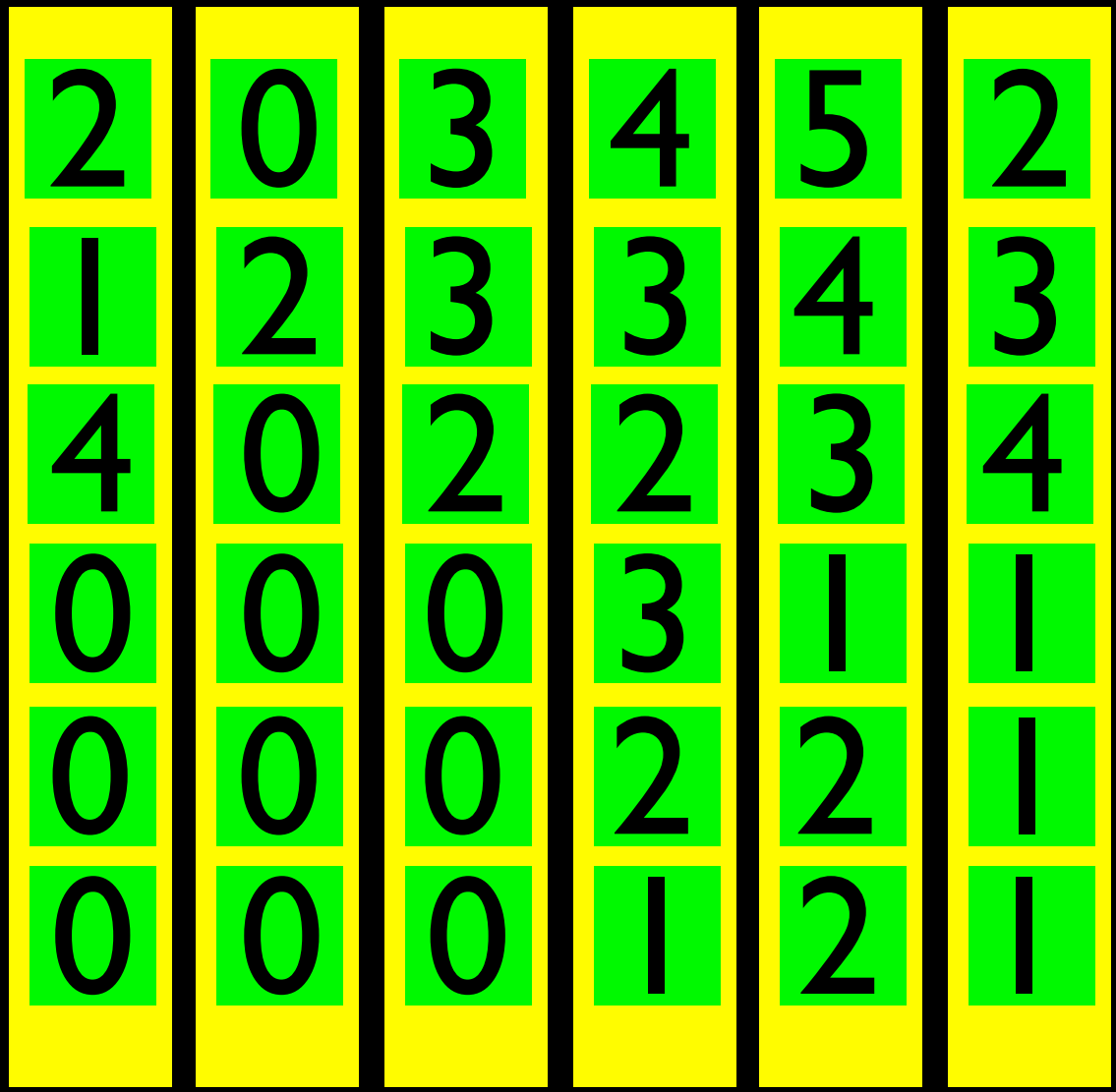


row reduction: subtract

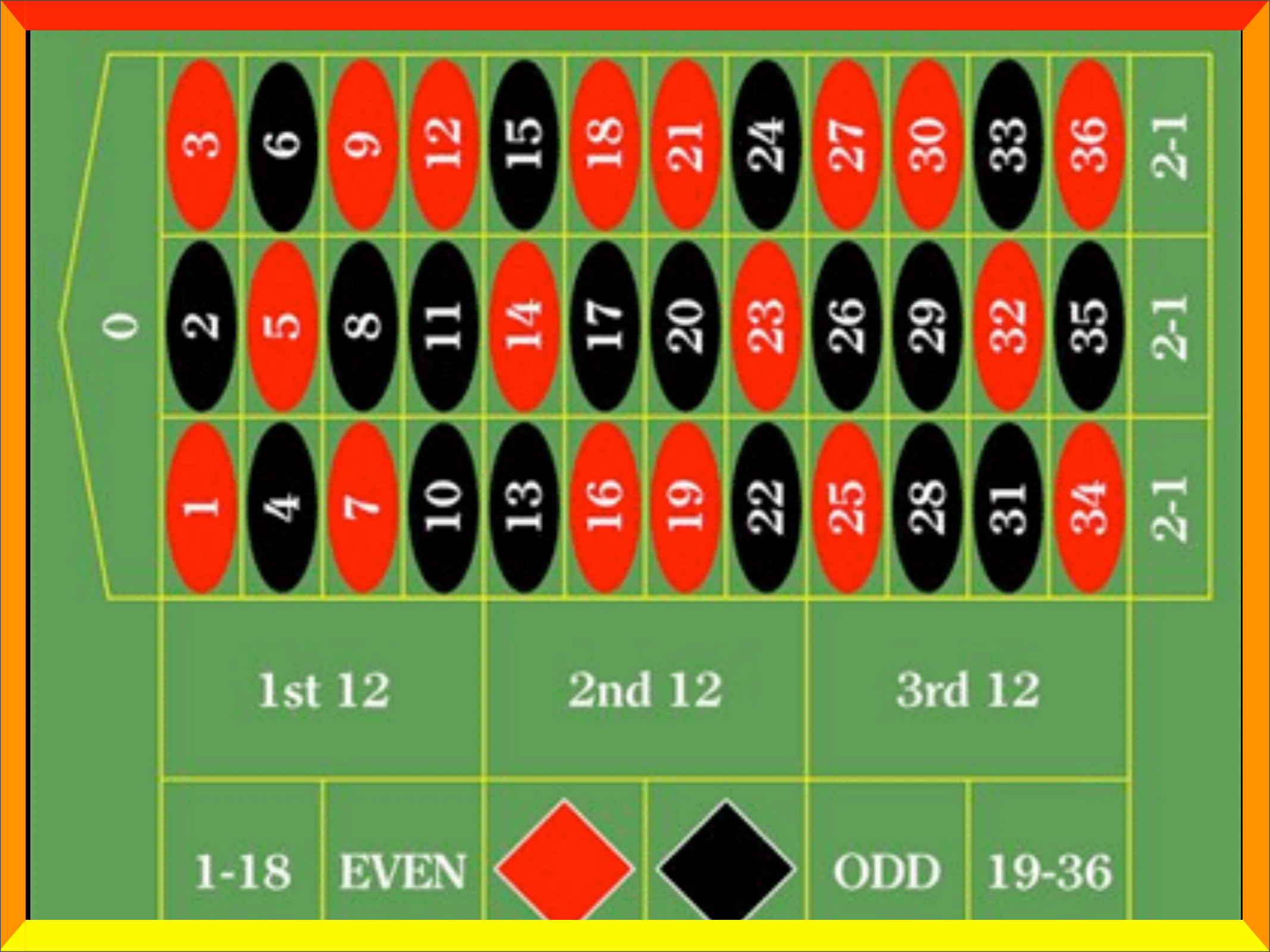


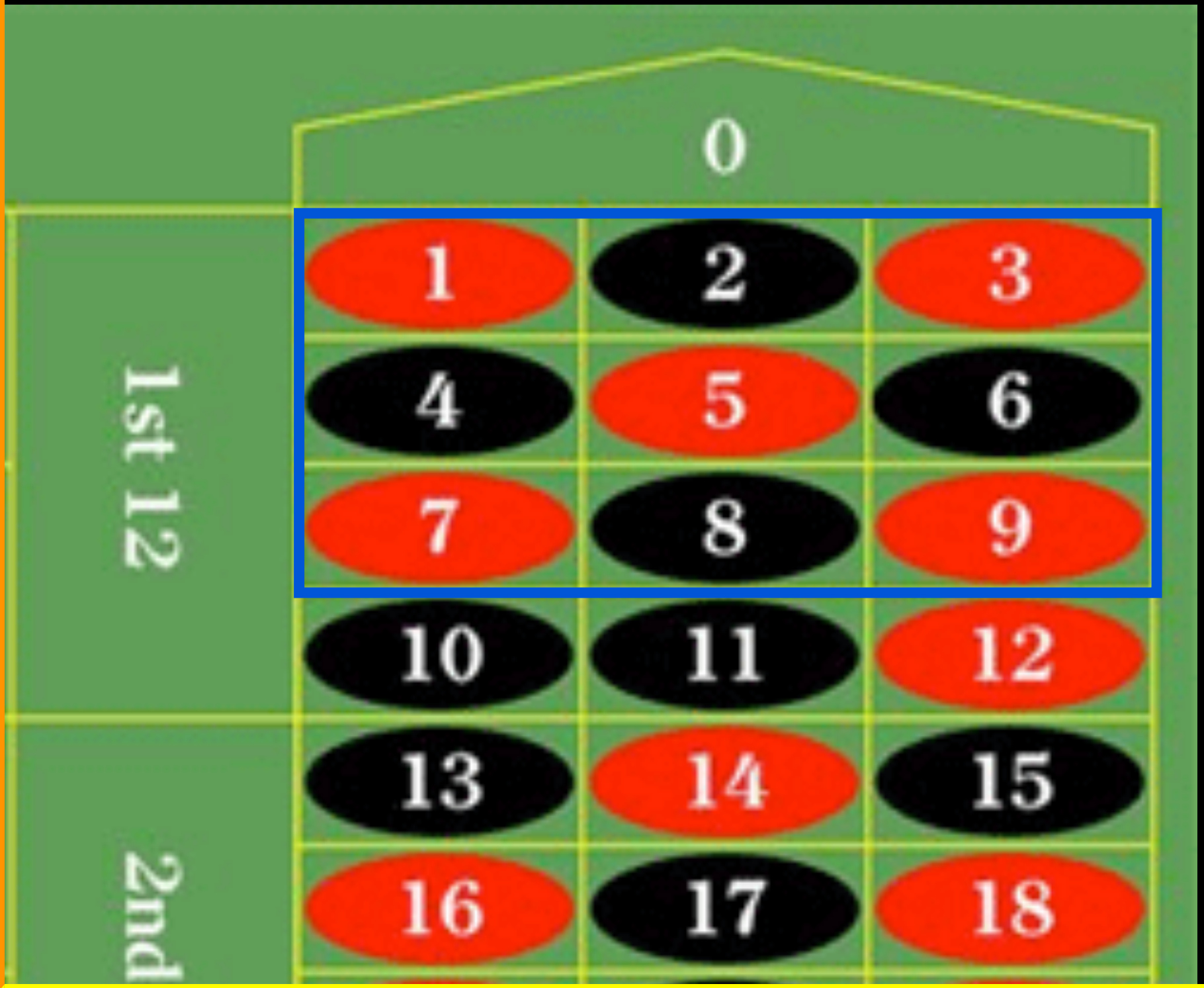

blackboard
problem

What is the determinant of



CASINO ROYALE
007
DVD
to win
CASINO ROYALE


0
3 6 9 12 15 18 21 24 27 30 33 36 2-1
2 5 8 11 14 17 20 23 26 29 32 35 2-1
1 4 7 10 13 16 19 22 25 28 31 34 2-1
1st 12
2nd 12
3rd 12
1-18
EVEN
ODD
19-36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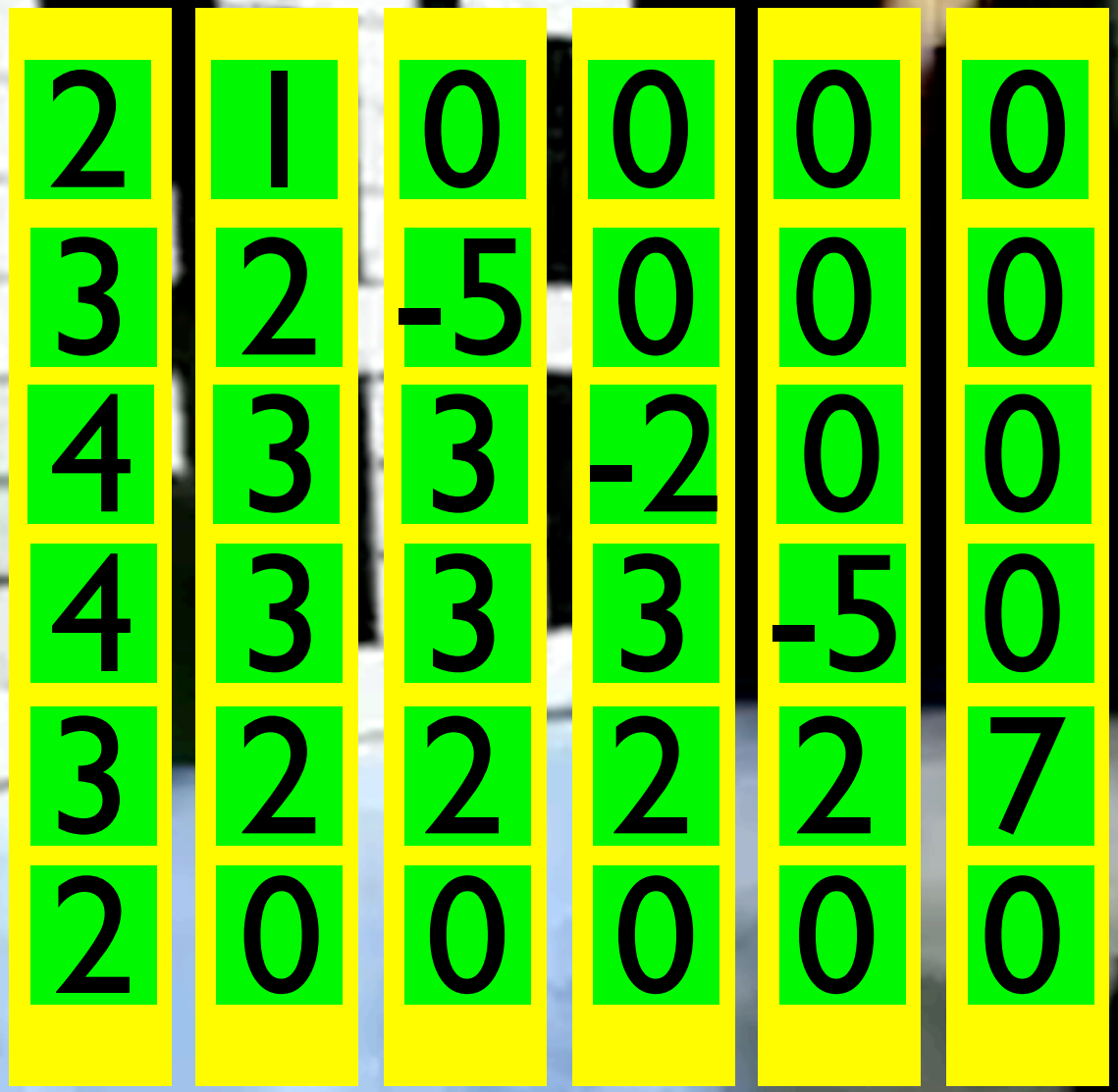
2nd

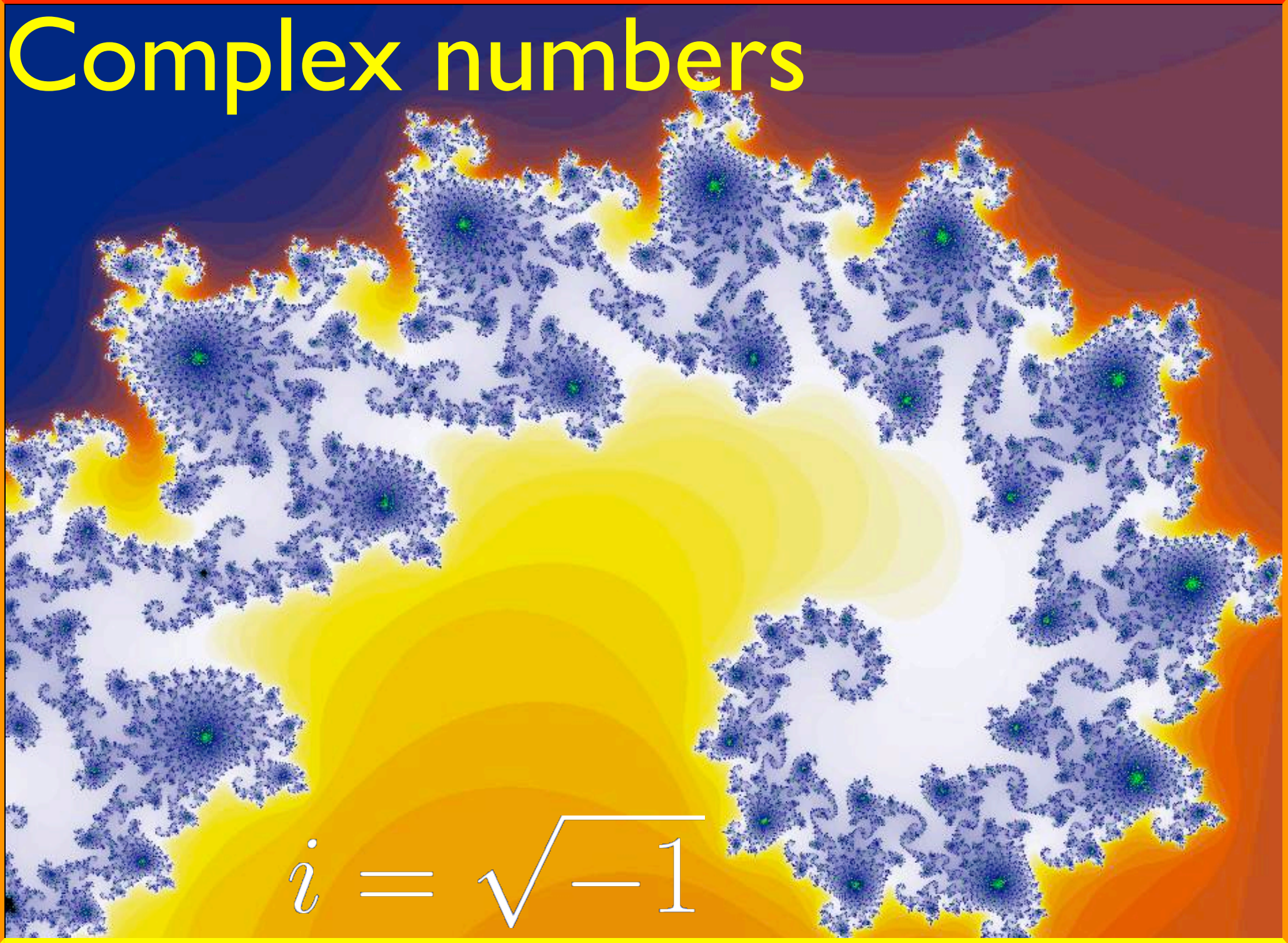
ready?
the next slide
gives the
problem

The video clip “4 min”
adds distraction and more
difficulty.

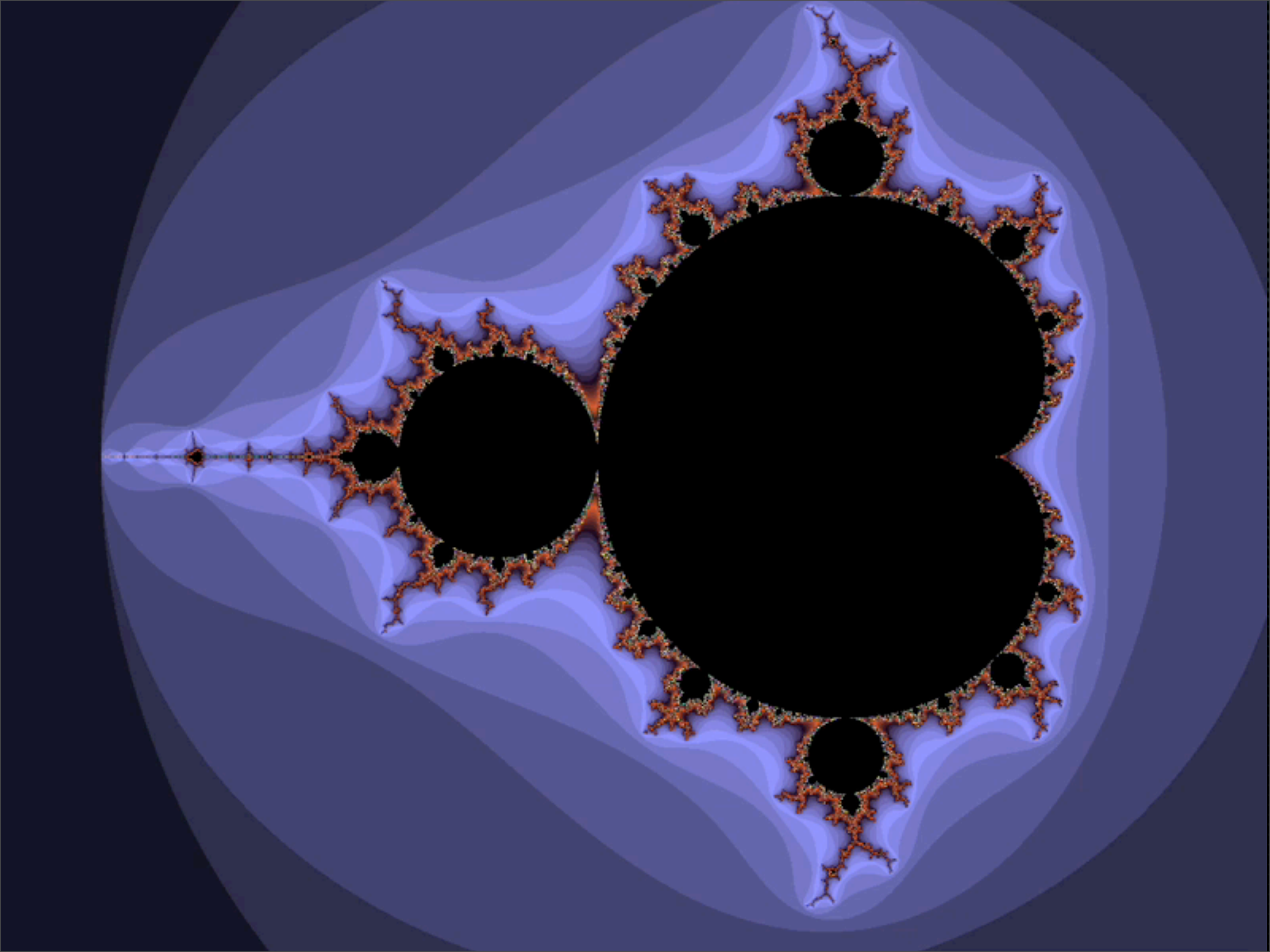
What is the determinant of



Complex numbers



$$i = \sqrt{-1}$$



Complex numbers

$$i = \sqrt{-1}$$

Gauss in 1825 : "The true metaphysics of the square root of -1 is elusive".

Euler Formula

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

Is the gateway to most secrets in complex numbers.

Proof:

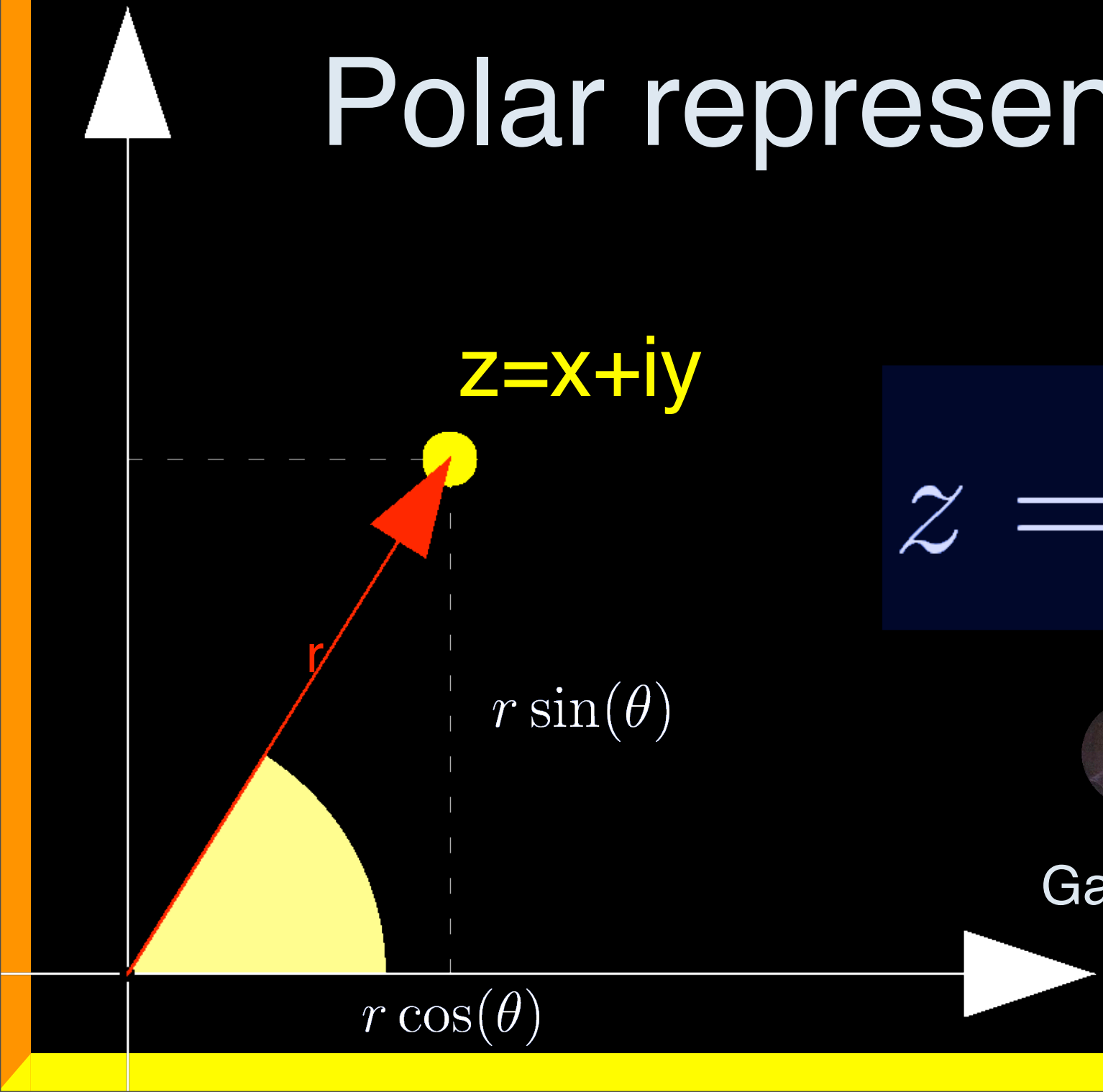
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

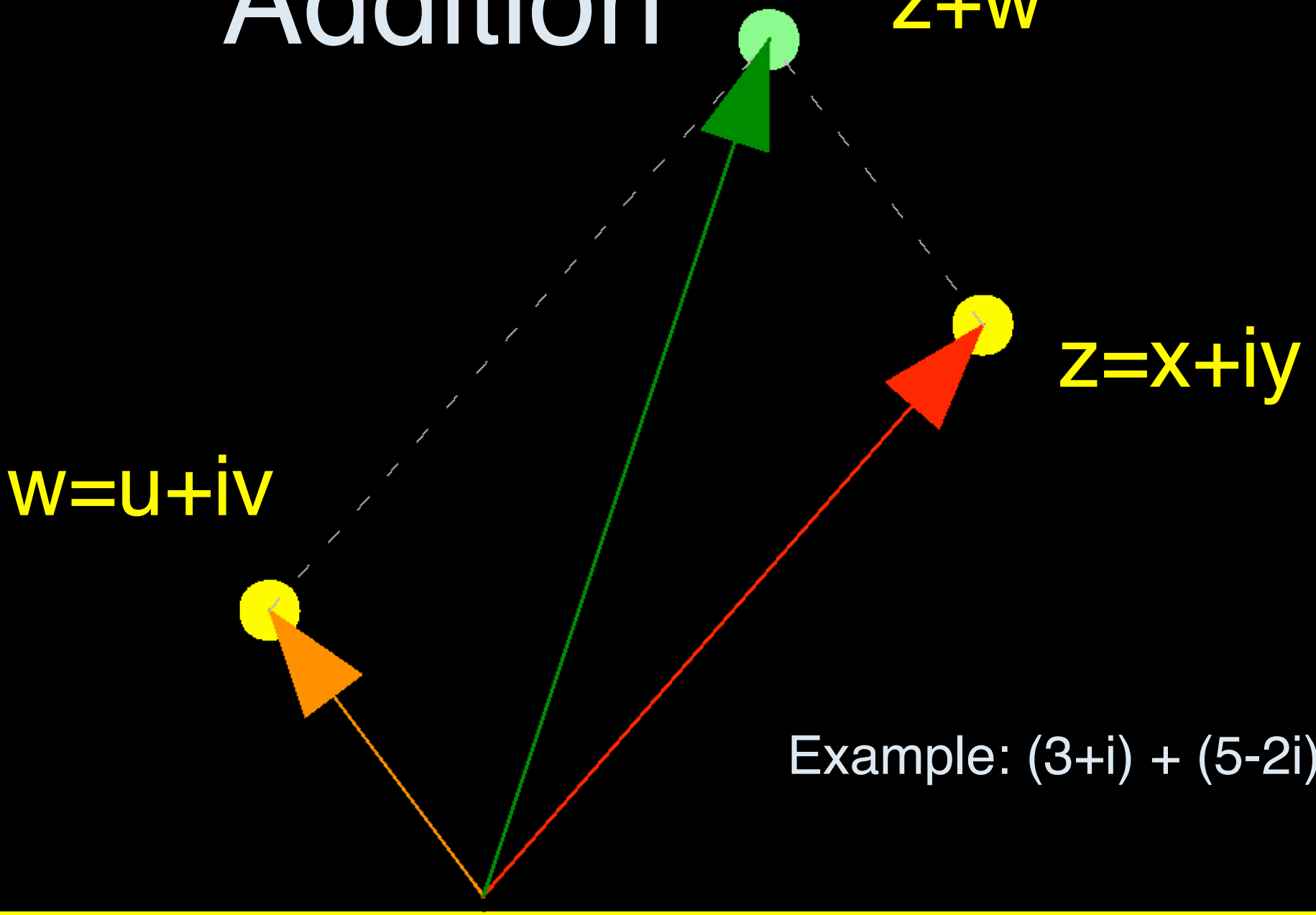
Polar representation



$$z = re^{i\theta}$$

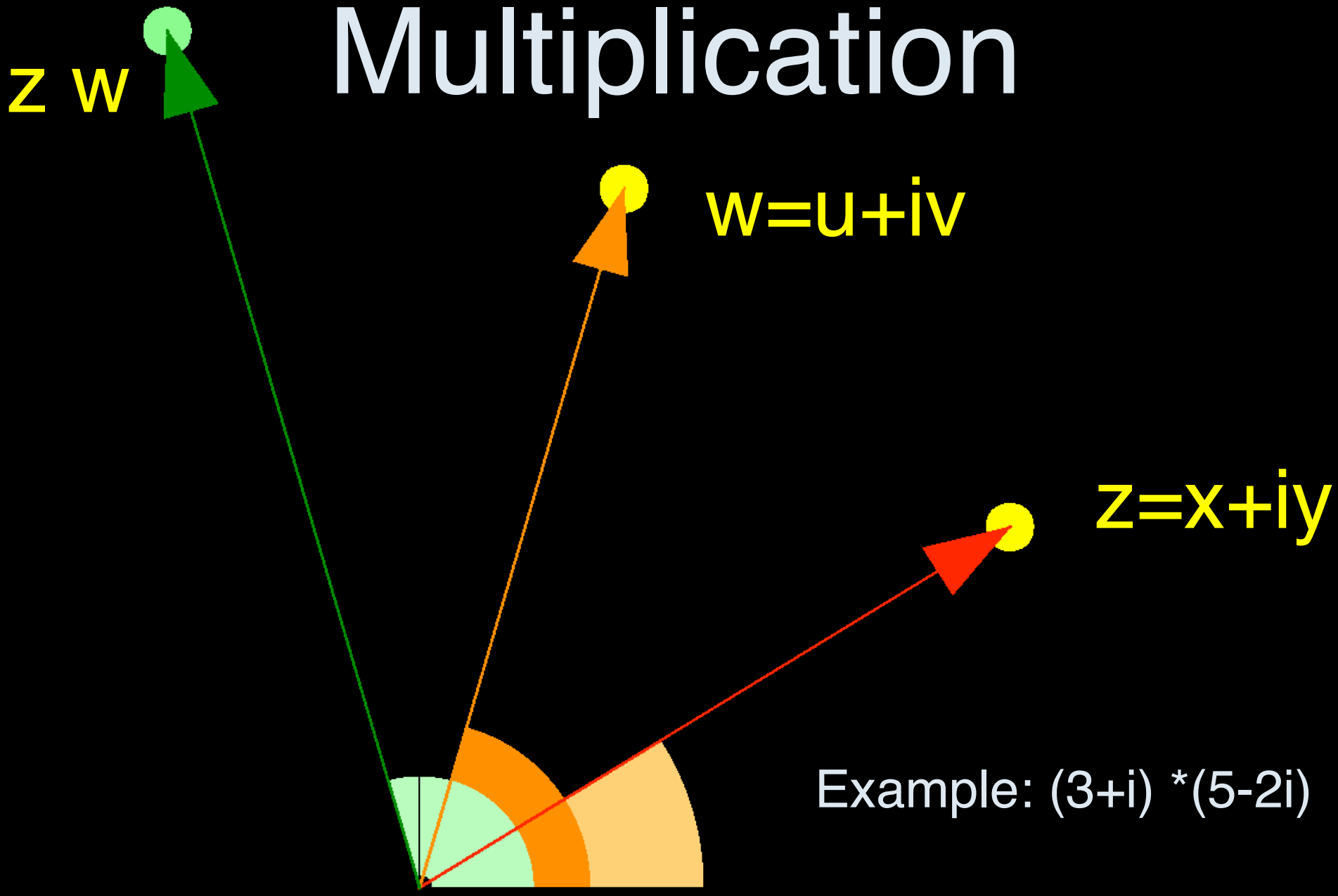
Gauss Plane

Addition



Example: (3+i) + (5-2i)

Multiplication

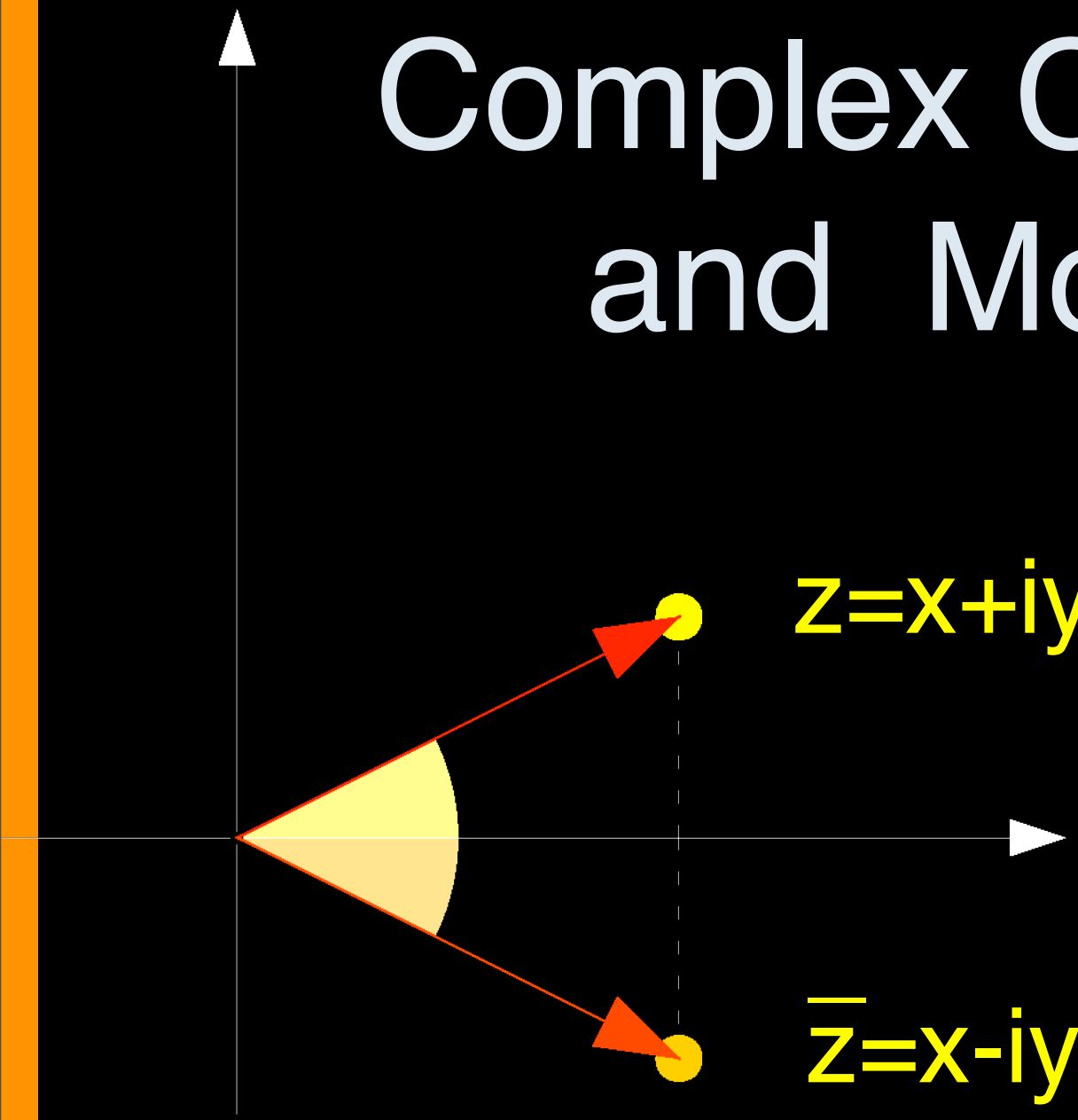


Example: (3+i) *(5-2i)

The most remarkable formula in math

$$1 + e^{i\pi} = 0$$

Complex Conjugate and Modulus

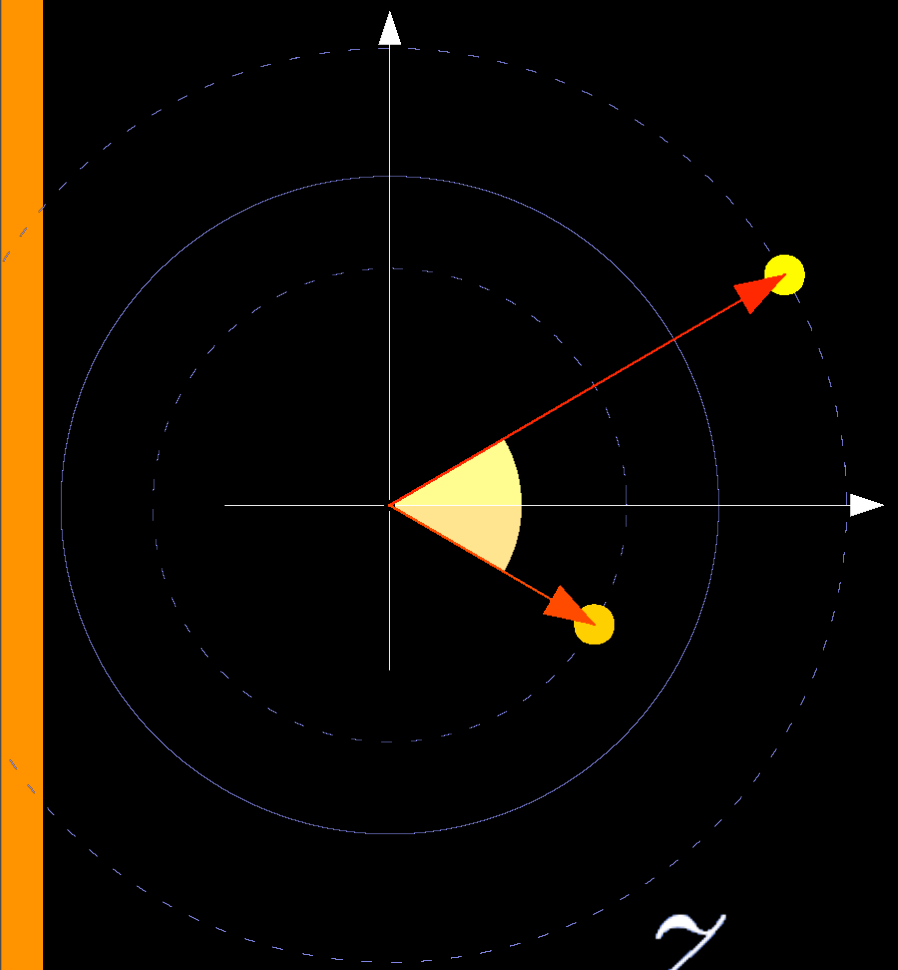


$z=x+iy$

$|z|^2=z\,\bar{z}$

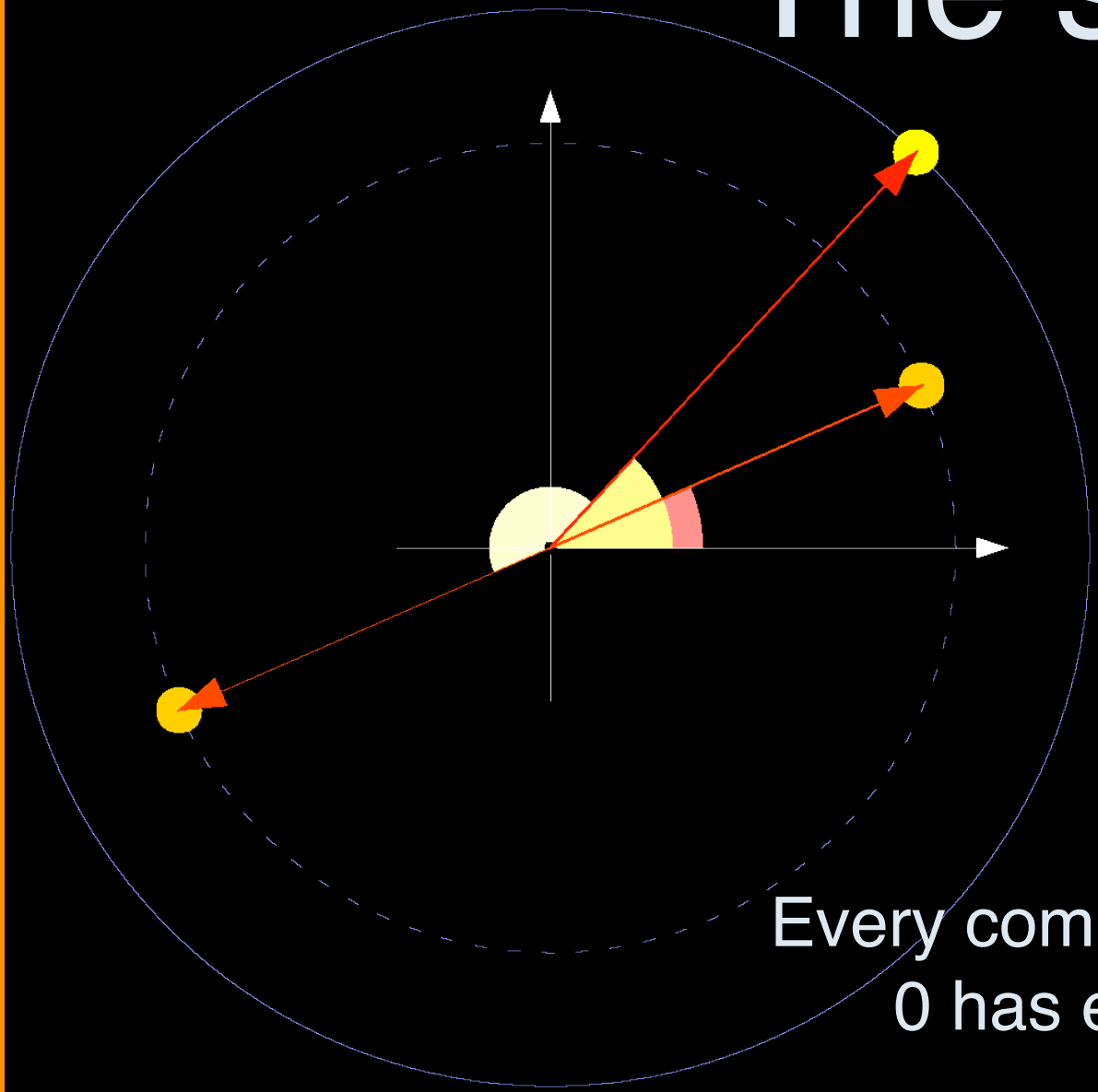
$\bar{z}=x-iy$

Division



$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

The square root

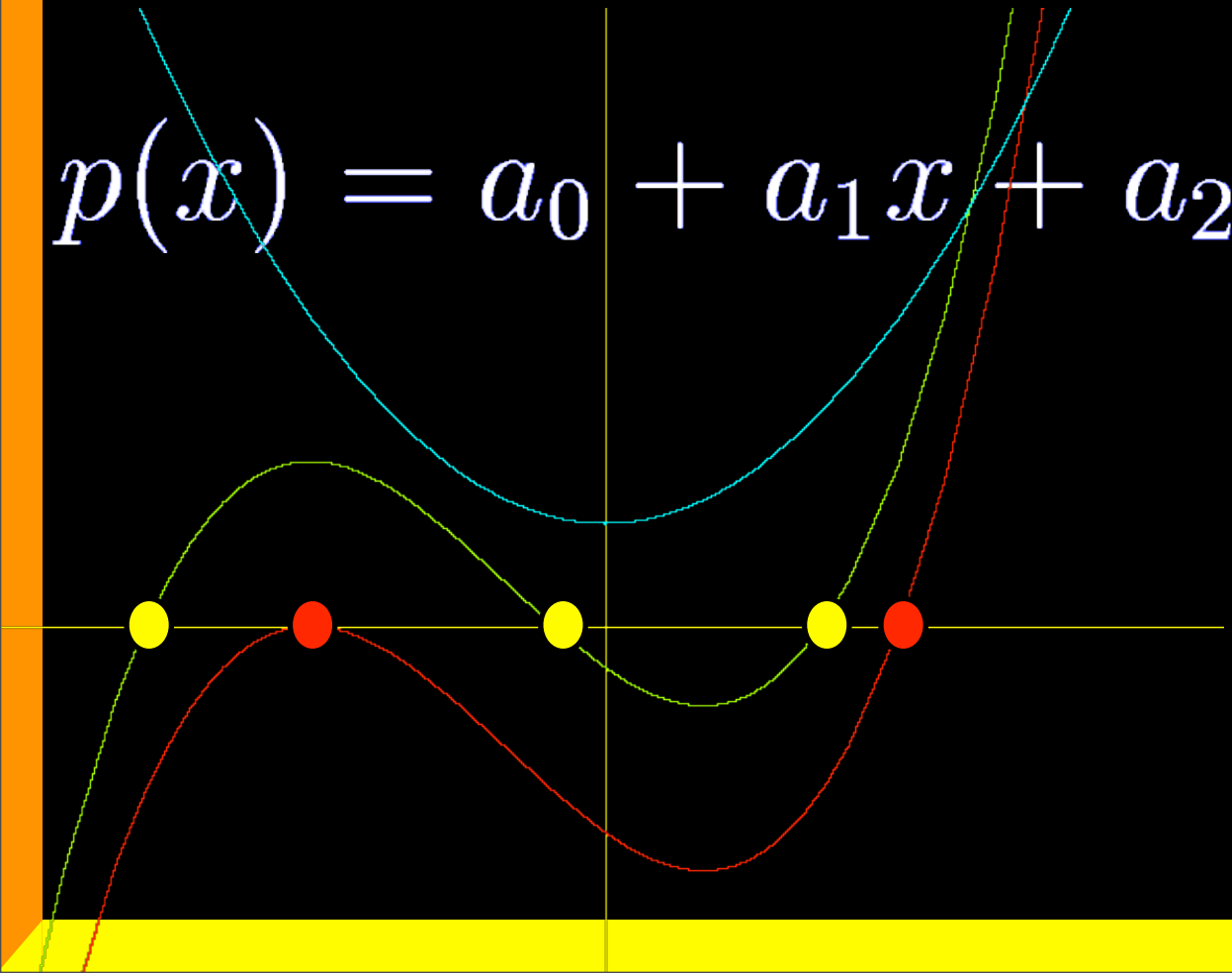


"Take square root of modulus and divide angle by 2".

Every complex number different from 0 has exactly 2 square roots.

Fundamental theorem of algebra

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$



A polynomial of degree n has exactly n roots p(x)=0

Eigenvalues

$$A\,v = \lambda v$$

Examples:

A	v is in the kernel
$A^T A = I$	v is in the rotation axes
$A^2 = I$	v is on the reflection space

Because det $(\lambda - A) = 0$, eigenvalues are the roots of the characteristic polynomial

$$A = \begin{matrix} a & b \\ c & d \end{matrix}$$

$$f(\lambda) = \lambda^2 - tr(A)\lambda + det(A)$$

$$\lambda_+ = \frac{tr(A) + \sqrt{tr(A)^2 - 4det(A)}}{2}$$

$$\lambda_- = \frac{tr(A) - \sqrt{tr(A)^2 - 4det(A)}}{2}$$

algebraic multiplicity

number of simultaneous roots of λ.

geometric multiplicity

dimension of $\ker(A-\lambda I)$

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$f_A(\lambda) = (\lambda-a)(\lambda-a)$$

algebraic multiplicity: 2
geometric multiplity: 1

Some good things to know

determinant

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

trace

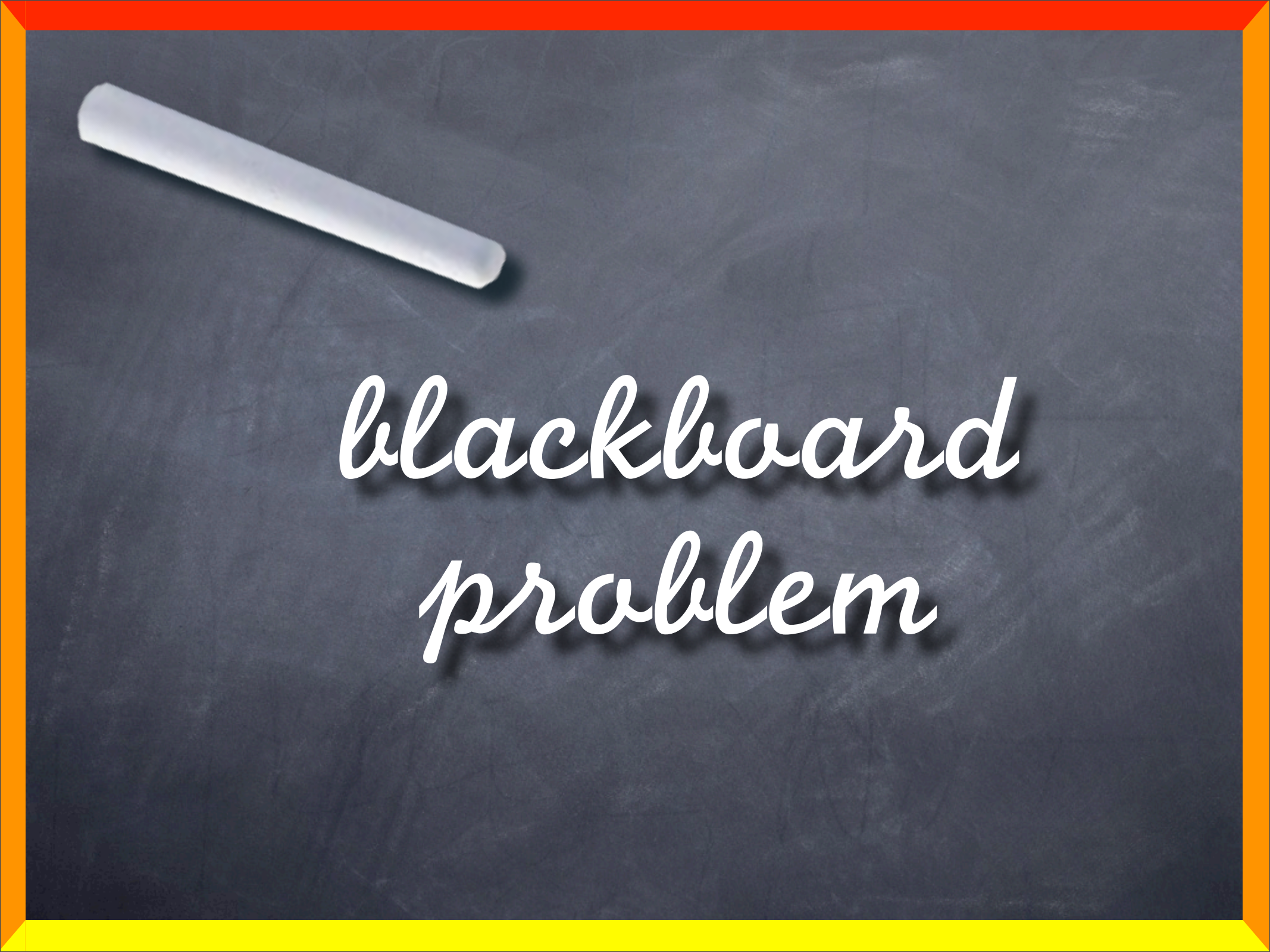
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

determinant of power

$$\det(A^k) = \lambda_1^k \lambda_2^k \cdots \lambda_n^k$$

trace of power

$$\operatorname{tr}(A^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$$


blackboard
problem

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Eigenvectors

$$Av=\lambda v$$
$$(A-\lambda I)v=0$$

v is in the kernel of $A-\lambda I$

The eigenspace is a kernel. If all eigenvalues are different, we have one eigenvector for each eigenvalue.

Example: $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$\lambda_+ = a+ib$ $\quad v_+ = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_- = a-ib$ $\quad v_- = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

All you have to do to find the eigenvalues is

compute kernels!

Example: $A = \begin{bmatrix} 1 & 2 \\ 20 & 3 \end{bmatrix}$

$\lambda_+ = 23$ $\quad v_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_- = -9$ $\quad v_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

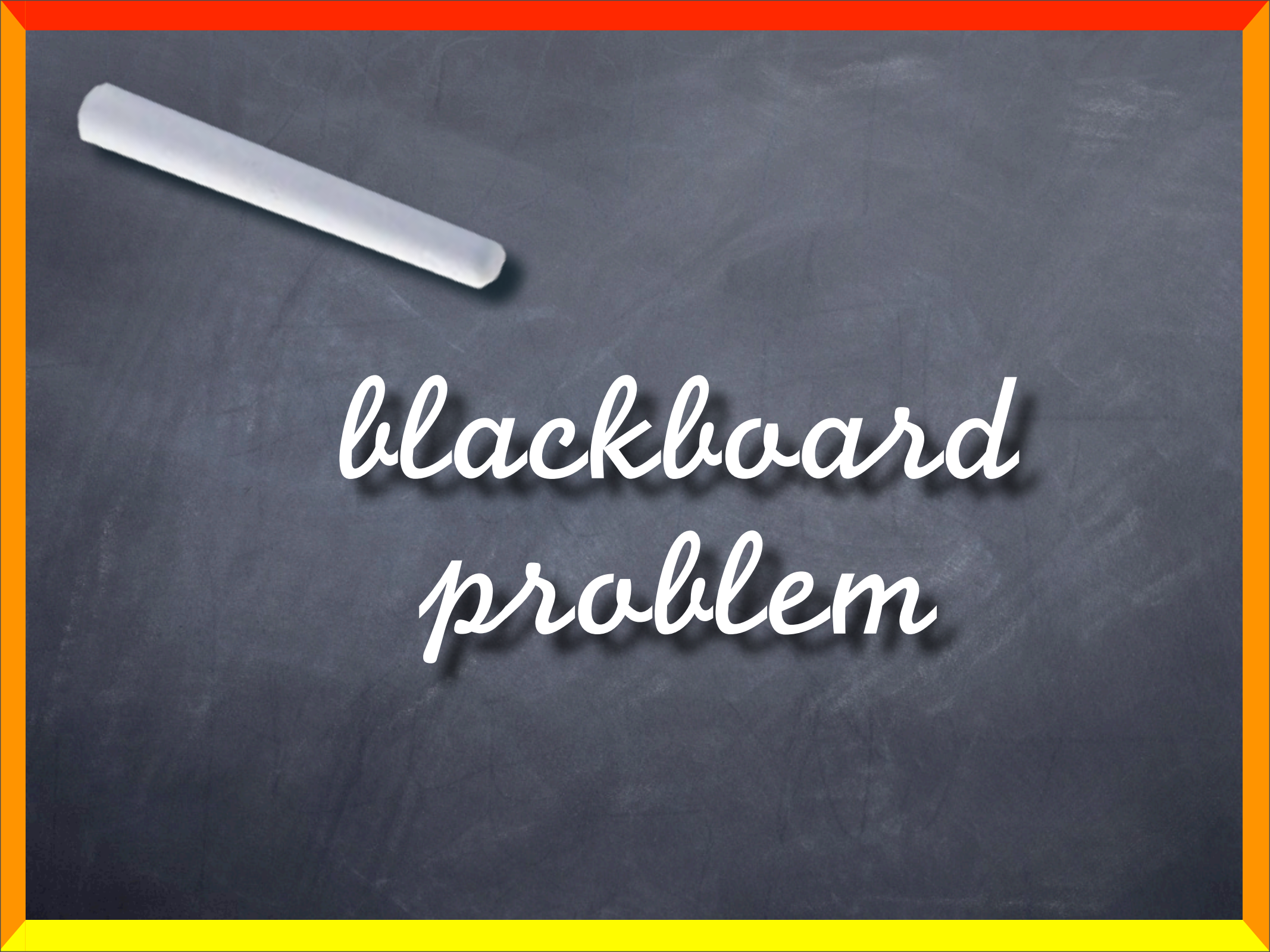
Example: (regular transition matrix)

$$A = \begin{pmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{pmatrix}$$

$\lambda_+ = 1$ A^T has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_- = 3/10$ from trace

A^{100} has eigenvalues 1 and $(3/10)^{100}$


blackboard
problem

Find all the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 5 \end{bmatrix}$$

Discrete dynamical systems

$$x(t+1) = A\,x(t)$$

has solution:

$$x(n) = A^{n}x(0)$$

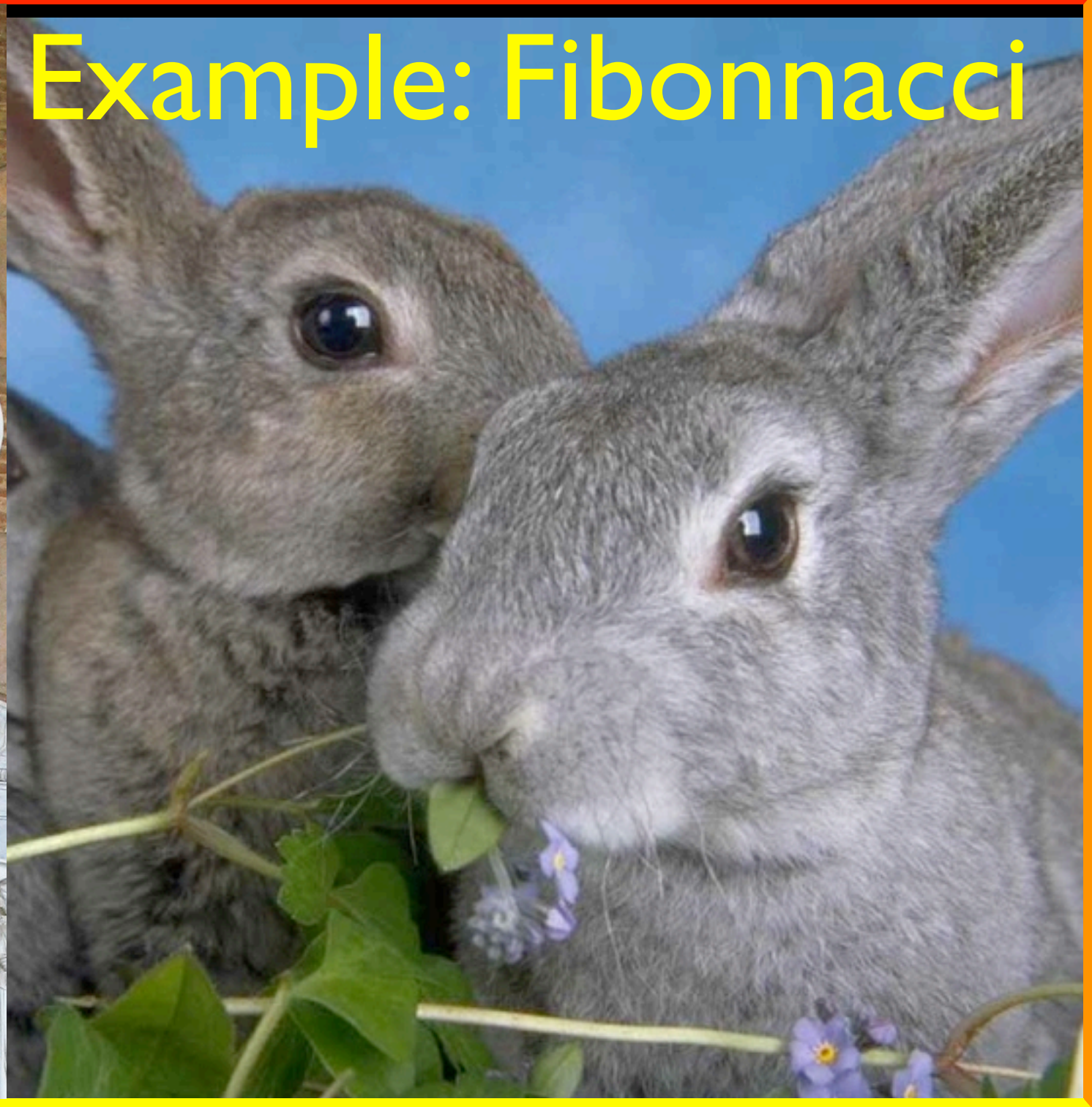
but this does not give any insight.

$$x(0) = a_1 v_1 + \ldots + a_n v_n$$

$$A v_k = \lambda_k v_k$$

$$x(t) = a_1 \lambda_1^t v_1 + \ldots + a_n \lambda_n^t v_n$$

Example: Fibonnacci



$$x(n+1) = x(n) + x(n-1) \qquad 1,1,2,3,5,8,13,21,\ldots$$

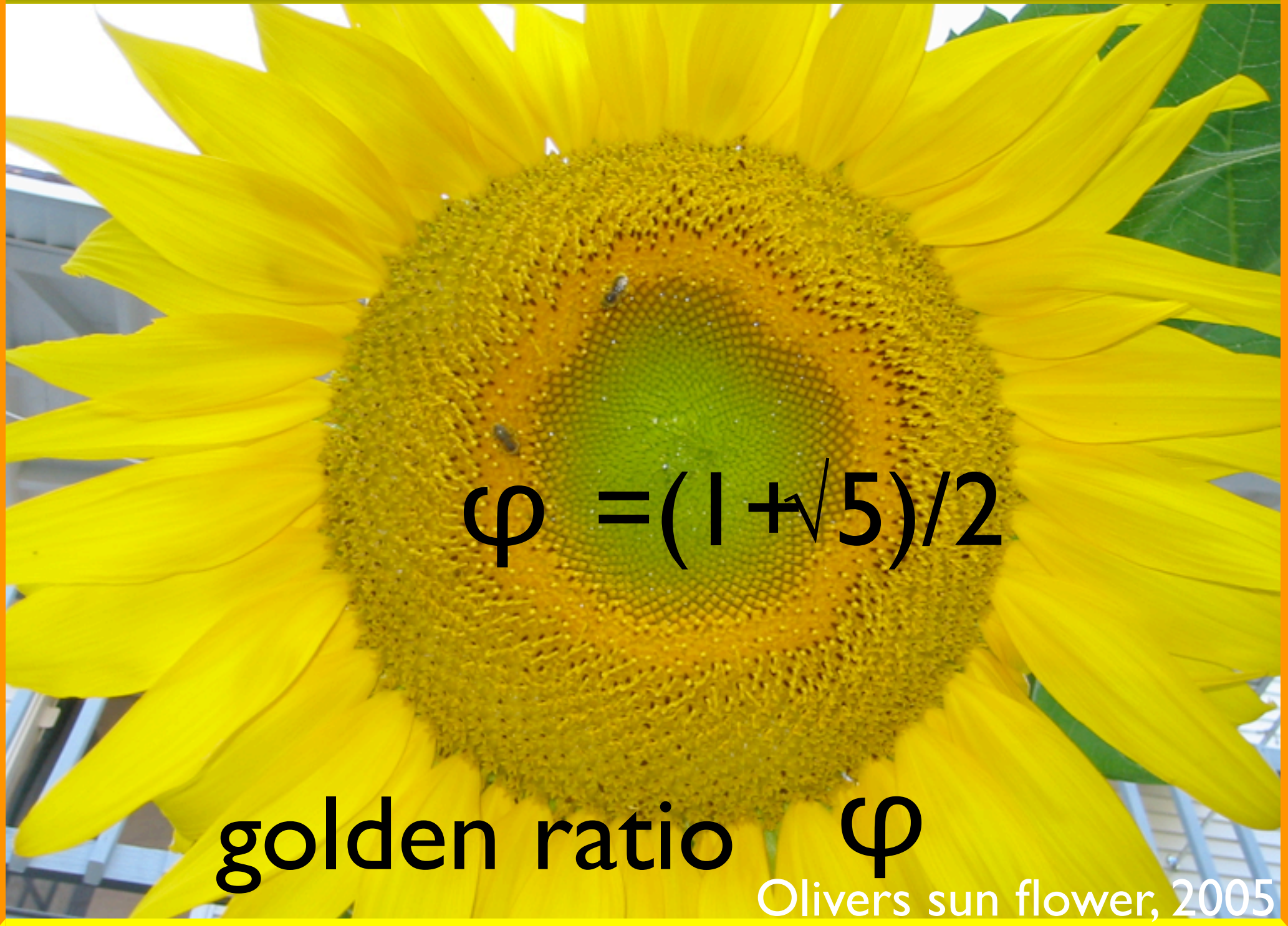
$$\begin{pmatrix} x(n+1) \\ x(n) \end{pmatrix} = \begin{pmatrix} x(n)+x(n-1) \\ x(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(n) \\ x(n-1) \end{pmatrix}$$

characteristic polynomial: $\lambda^2 - \lambda - 1 = f_A(\lambda)$

eigenvalues=$\varphi, 1-\varphi$ eigenvectors= $\begin{pmatrix} \varphi \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1-\varphi \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \begin{pmatrix} 1-\varphi \\ 1 \end{pmatrix}$$

$$x(n) = (\varphi^n - (1-\varphi)^n)/\sqrt{5}$$

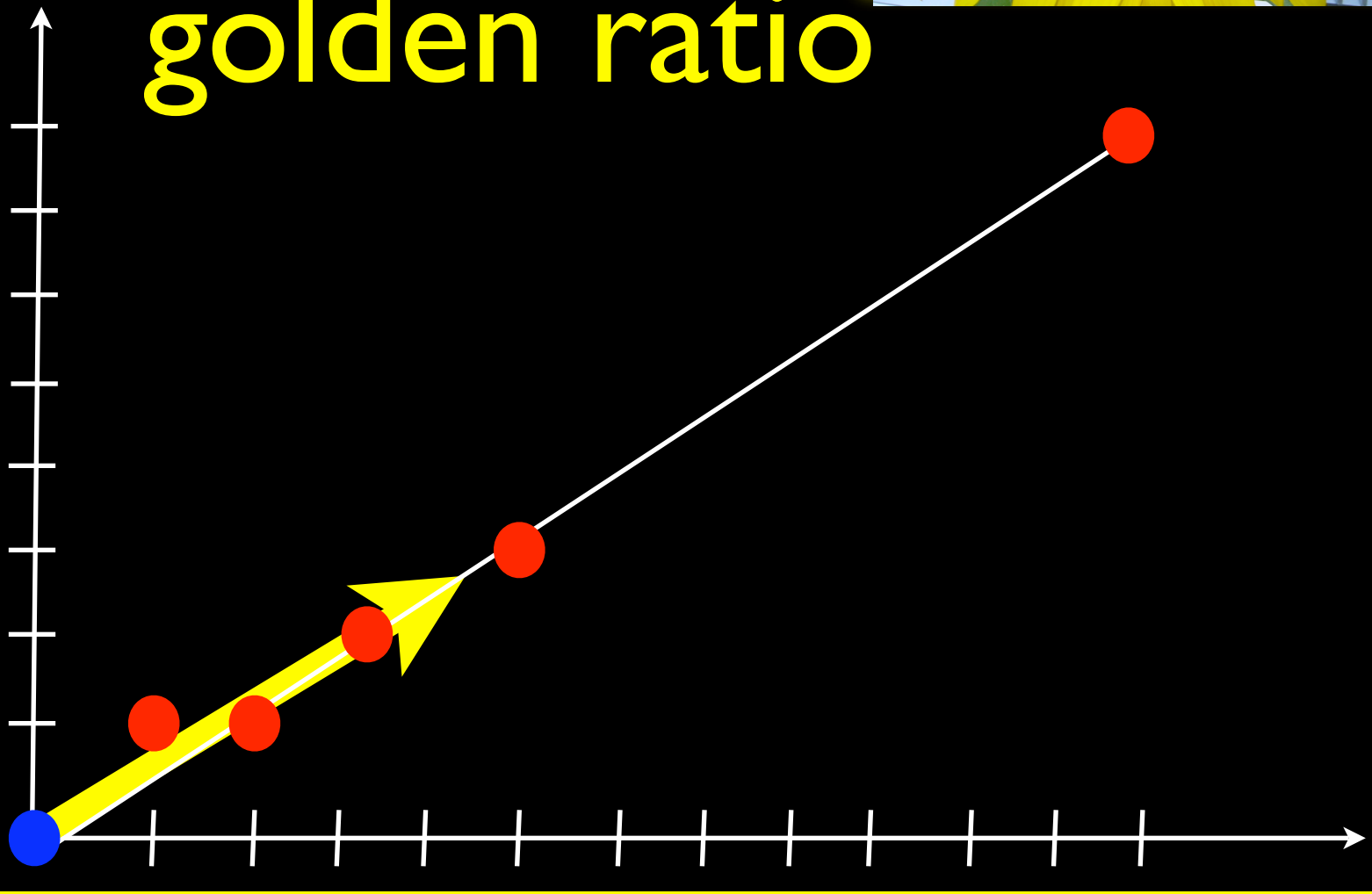

φ =(1+√5)/2
golden ratio φ
Olivers sun flower, 2005

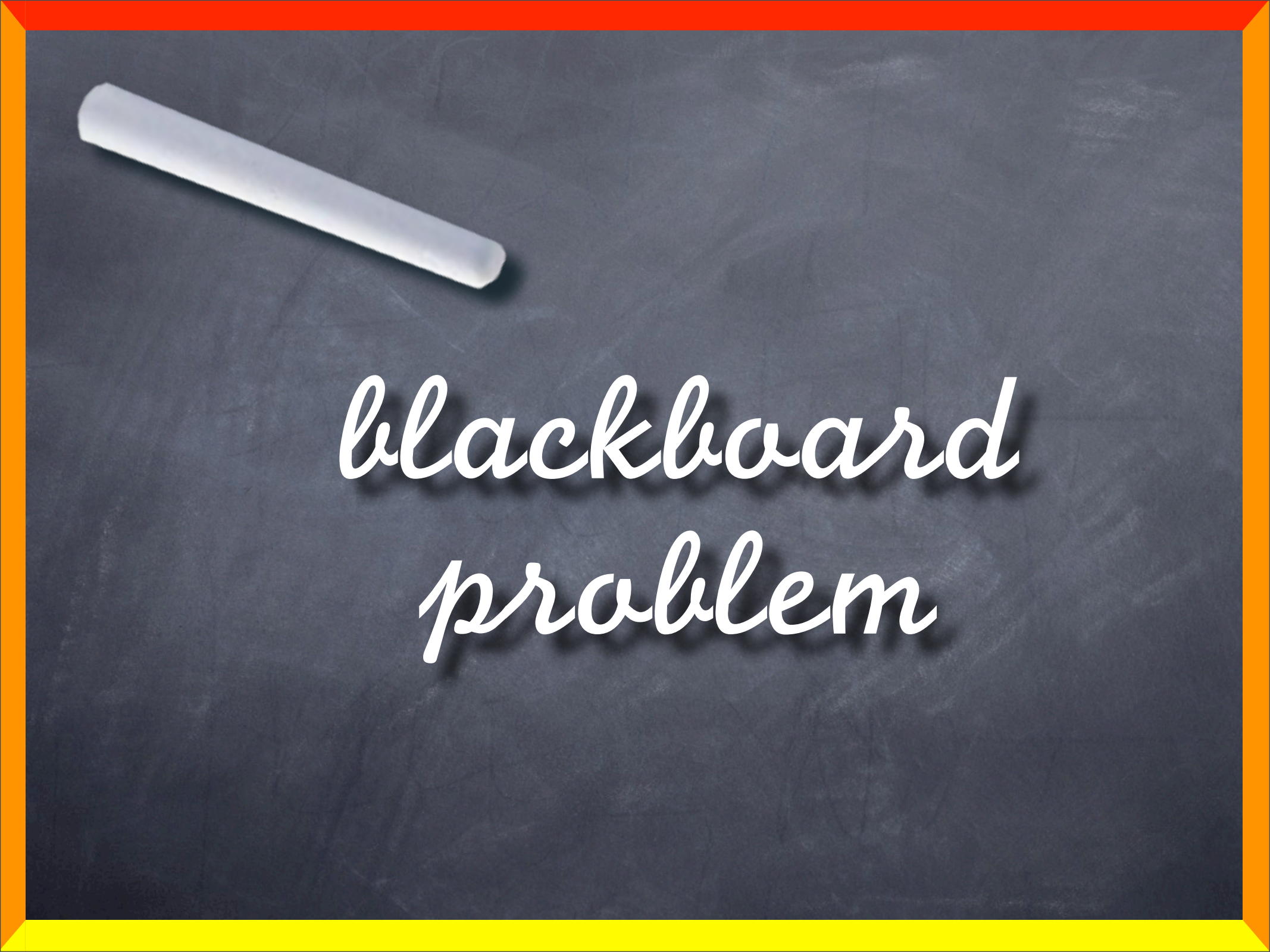
$$\varphi = \frac{\sqrt{5}+1}{2} = 1.616...$$

$1/\varphi = \varphi - 1 = 0.616...$

golden ratio

2/1
3/2
5/3
8/5
13/8
21/13
...
φ

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problem

The Lilac Bush problem

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

new branches
old branches

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

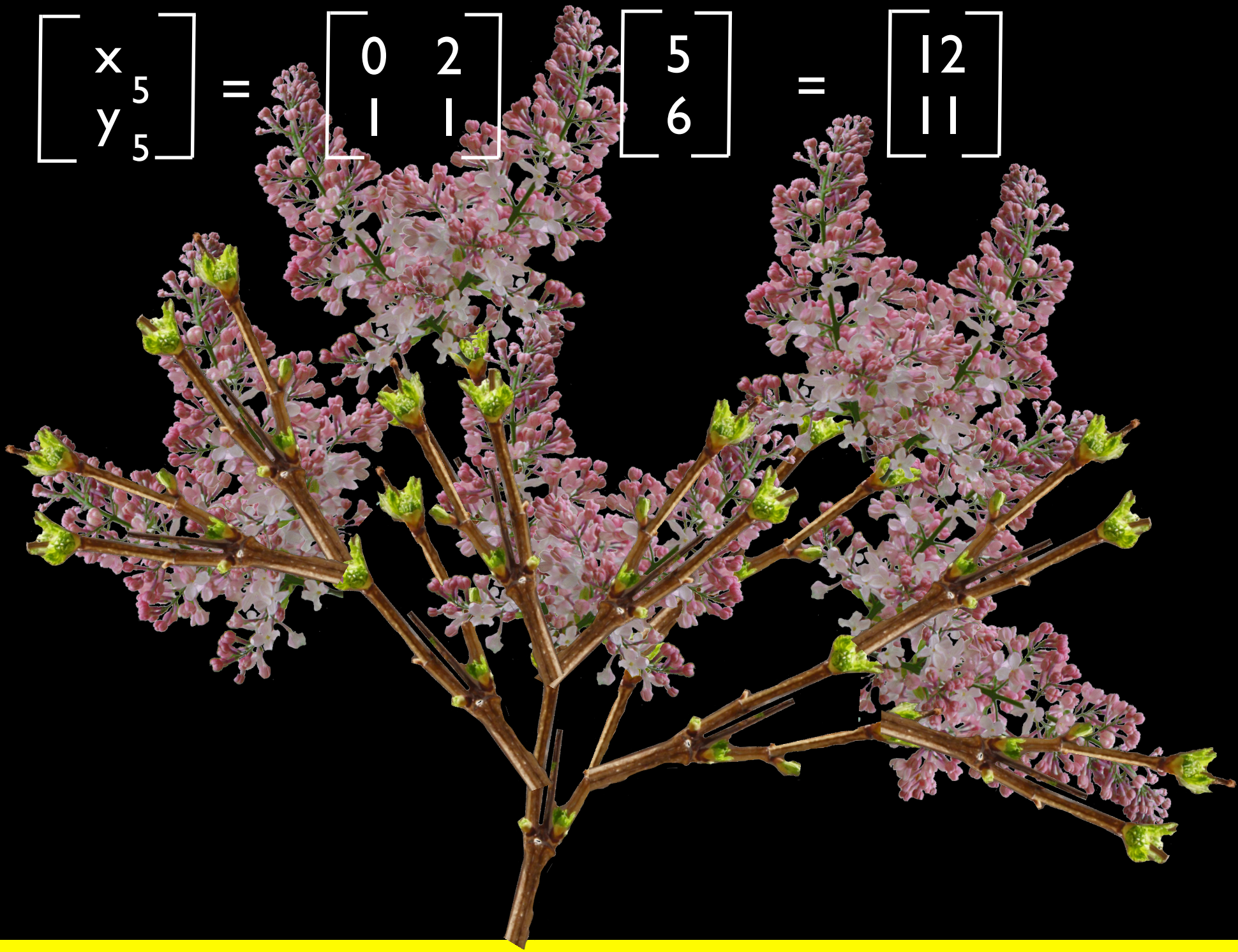
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

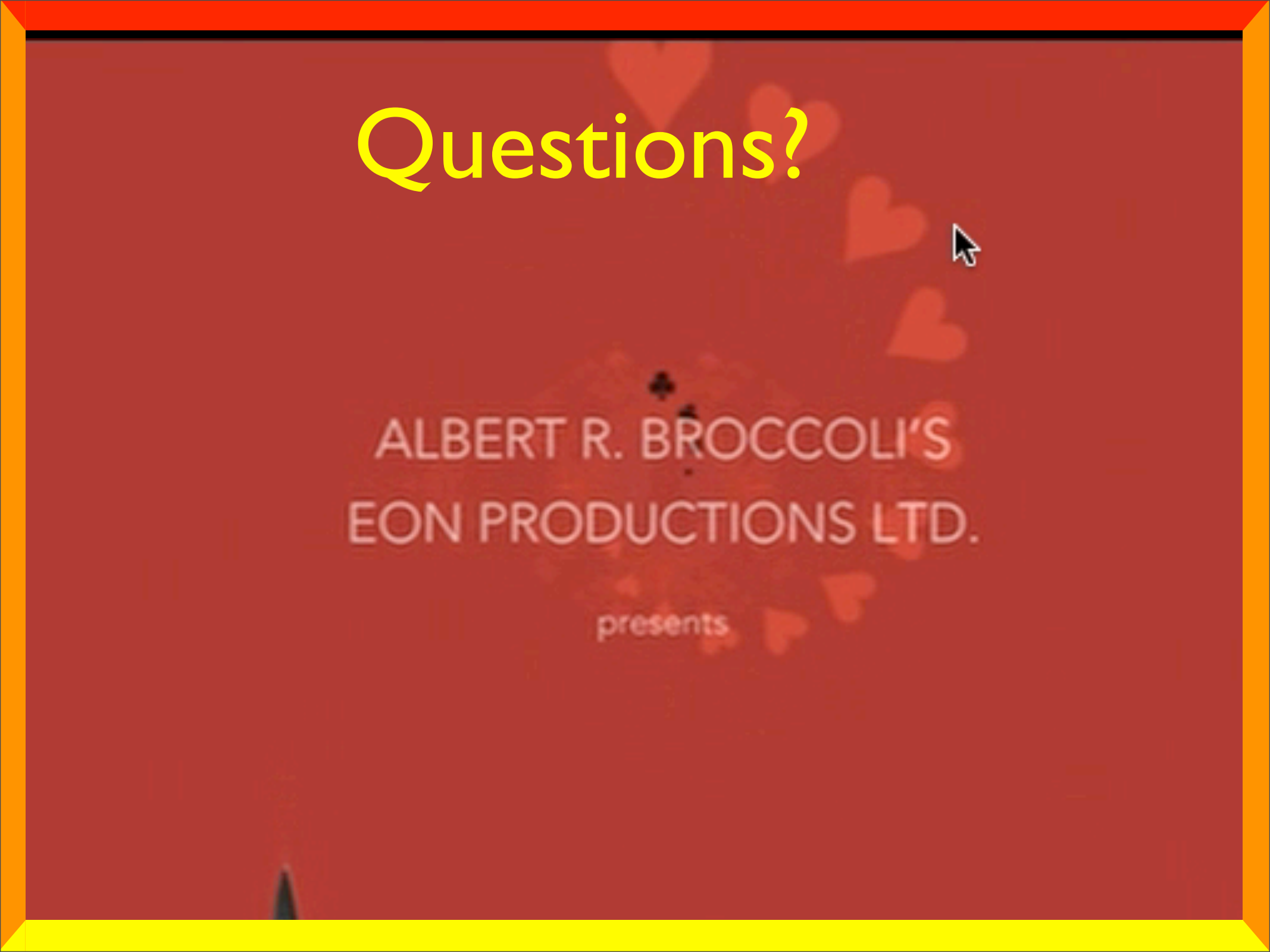
$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$



Diagonalization

If all eigenvalues of a matrix A are different, then A can be diagonalized.

Example:

$$\begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 4 \end{bmatrix}$$

Questions?