

Math21b

Review to first midterm

Spring 2008

Oliver Knill, March 2, 2008

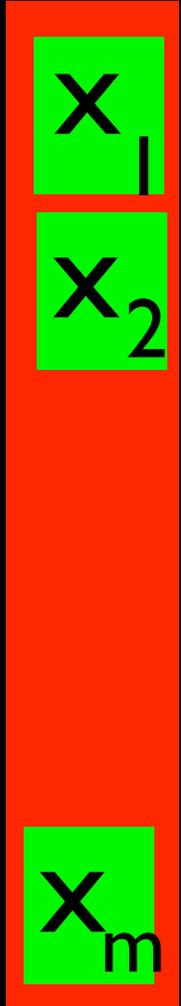
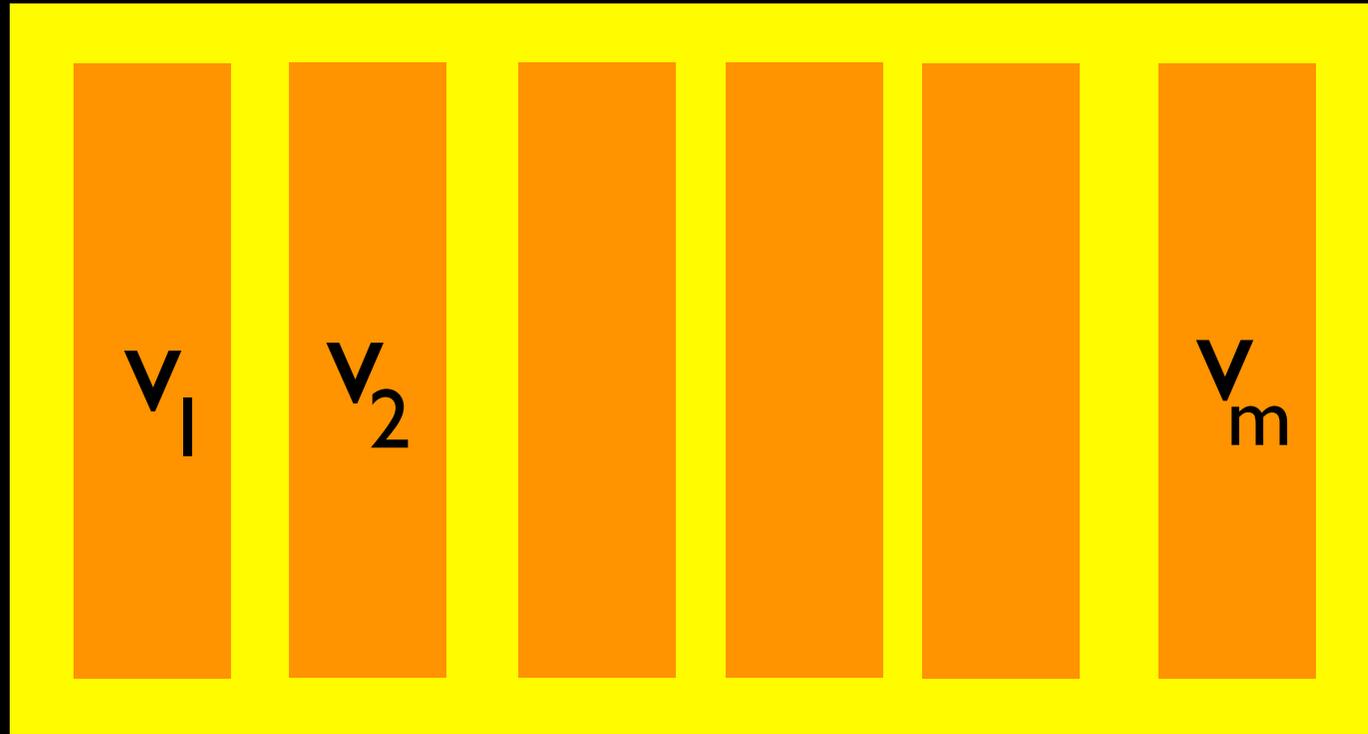
A pocket watch with a white face and black Roman numerals is mounted on a wooden structure. The watch has a gold-colored case and a blue metal cap. The hands are gold and the watch is set against a background of light-colored wood with dark wood accents.

Math 2 I b
Spring 2008

Matrices

column picture

$$Ax =$$



$$= x_1 v_1 + x_2 v_2 + \dots + x_m v_m$$

The equation shows the expansion of the matrix-vector product. It consists of a green box with x_1 followed by an orange bar with v_1 , a plus sign, a green box with x_2 followed by an orange bar with v_2 , an ellipsis, another plus sign, a green box with x_m followed by an orange bar with v_m .

row picture

$$Ax =$$



Example: $Ax = 0$, means x is perpendicular to row space.

Example: $Ax = b$, means b_k is the dot product of the k 'th row with x .

Example

to find all vectors
perpendicular to these
two vectors:

1
2
4
6

1
4
6
8

1 2 4 6

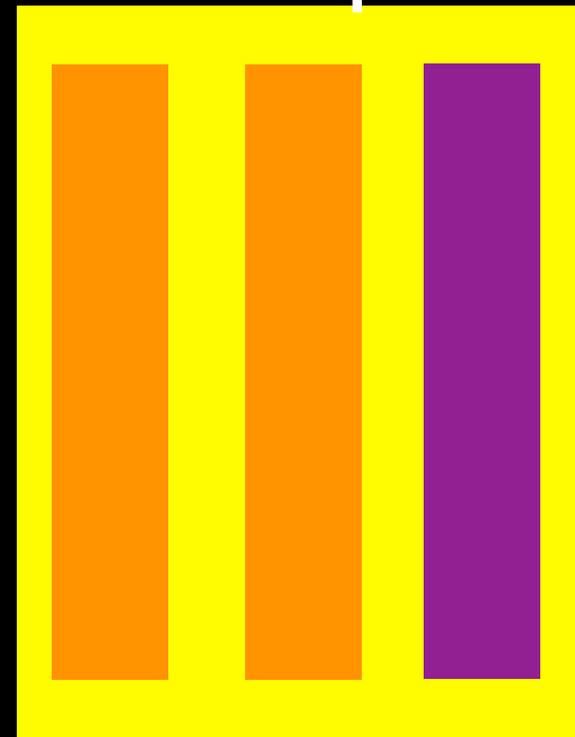
1 4 6 8

solve $Ax = 0$, where A contains the
vectors as rows.

$n \times m$

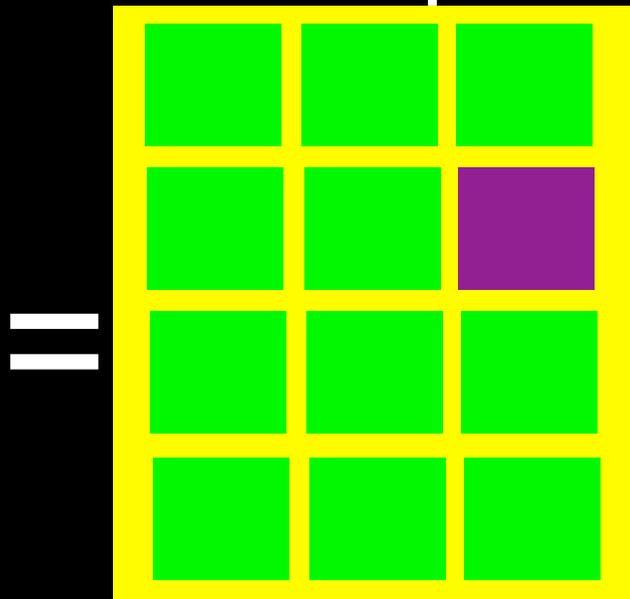


$m \times p$



\cdot

$n \times p$



$=$

matrix
multiplication

Problem:

Can we multiply a 4×5
matrix A

with a 5×4 matrix B ?

in other words:

Is AB defined?

Bonus: is BA defined?

Matrix algebra:

With $n \times n$ matrices A, B, C, D, \dots one can work as with numbers

$$A + B = B + A$$

$$a(A + B) = aA + aB$$

$$A(B + C) = AB + AC$$

$$A(B C) = (A B) C$$

etc

except for two things in general we have:

$$A B \neq B A$$

and

A^{-1} might not exist even for nonzero A

if A, B are invertible: $(A B)^{-1} = B^{-1} A^{-1}$

True or False?

A, B, arbitrary
n x n matrices

$$(A + B)(A - B) = A^2 - B^2$$

$$(I + A + A^2 + A^3) = (I - A^4)(I - A)^{-1}$$

assuming $(I - A)$ is invertible

$$I = I_n$$

Systems of linear equations

$$A x = b$$

$n \times n$ matrix A

exactly one solution

$$x = A^{-1} b$$

Matrix equation

$$A X = B$$

$$X = A^{-1} B$$

possibilities $Ax = b$



one solution
consistent

$$\text{rank}(A) = n$$



zero solutions
inconsistent

$$\text{rank}(B) > \text{rank}(A)$$



infinitely many
consistent



$$B = [A | b]$$

augmented matrix

How many solutions do we have if the augmented matrix is

1	4	1	6	2
0	0	1	2	3
0	0	0	1	1
0	0	0	0	0

Row reduction

Gauss-Jordan elimination

Scale
a
row

S

Swap
two
rows

S

Subtract
row
from
other
row

S

First
blackboard
problem ✓

Row reduce the X matrix

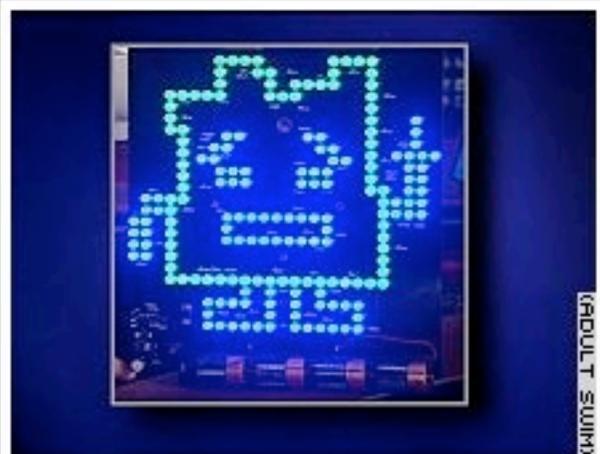
8	0	0	0	8
0	8	0	8	0
0	0	8	0	0
0	8	0	8	0
8	0	0	0	8

The story of the Boston milkshake scare:



Ad campaign triggers bomb scare in Boston

POSTED: 6:31 p.m. EST, January 31, 2007



Lightboards featuring a Mooninite character have been in place for weeks in 10 U.S. cities, a Turner Broadcasting statement says.

ADVERTISER LINKS

• [Women's Apparel](#)

STORY HIGHLIGHTS

- Packages were promotional material for Adult Swim network show
- Devices included a light board that displayed a character on the show
- The devices were placed in 10 cities across the country

Adjust font size:

BOSTON, Massachusetts (CNN) -- Electronic light boards featuring an adult-cartoon character triggered bomb scares around Boston on Wednesday, spurring authorities to close two bridges and a stretch of the Charles River before determining the devices were harmless.

Turner Broadcasting Co., the harmless magnetic lights air cartoon "Aqua Teen Hunger" displayed one of the Mooninite the show, greeting visitors w

"While the concern is lessen



Gotscha!



You can no more escape



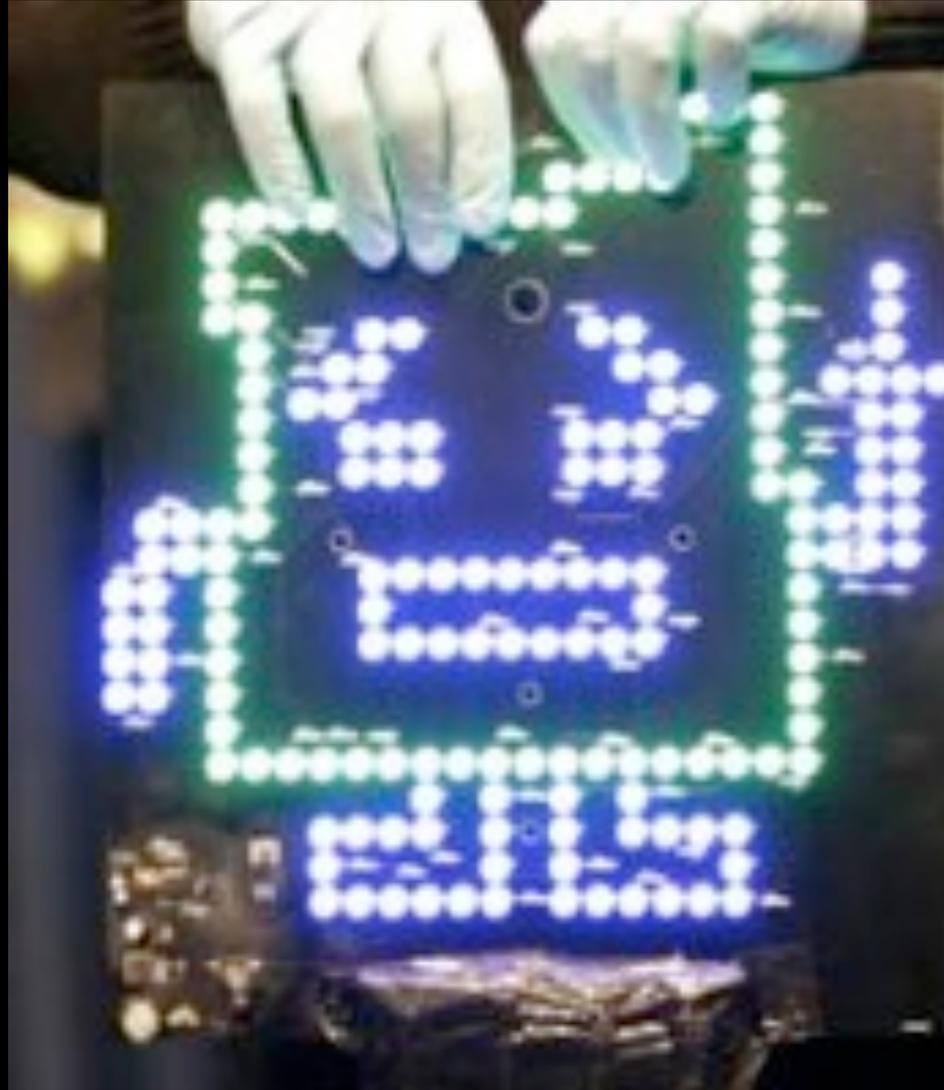
Executed with water gun



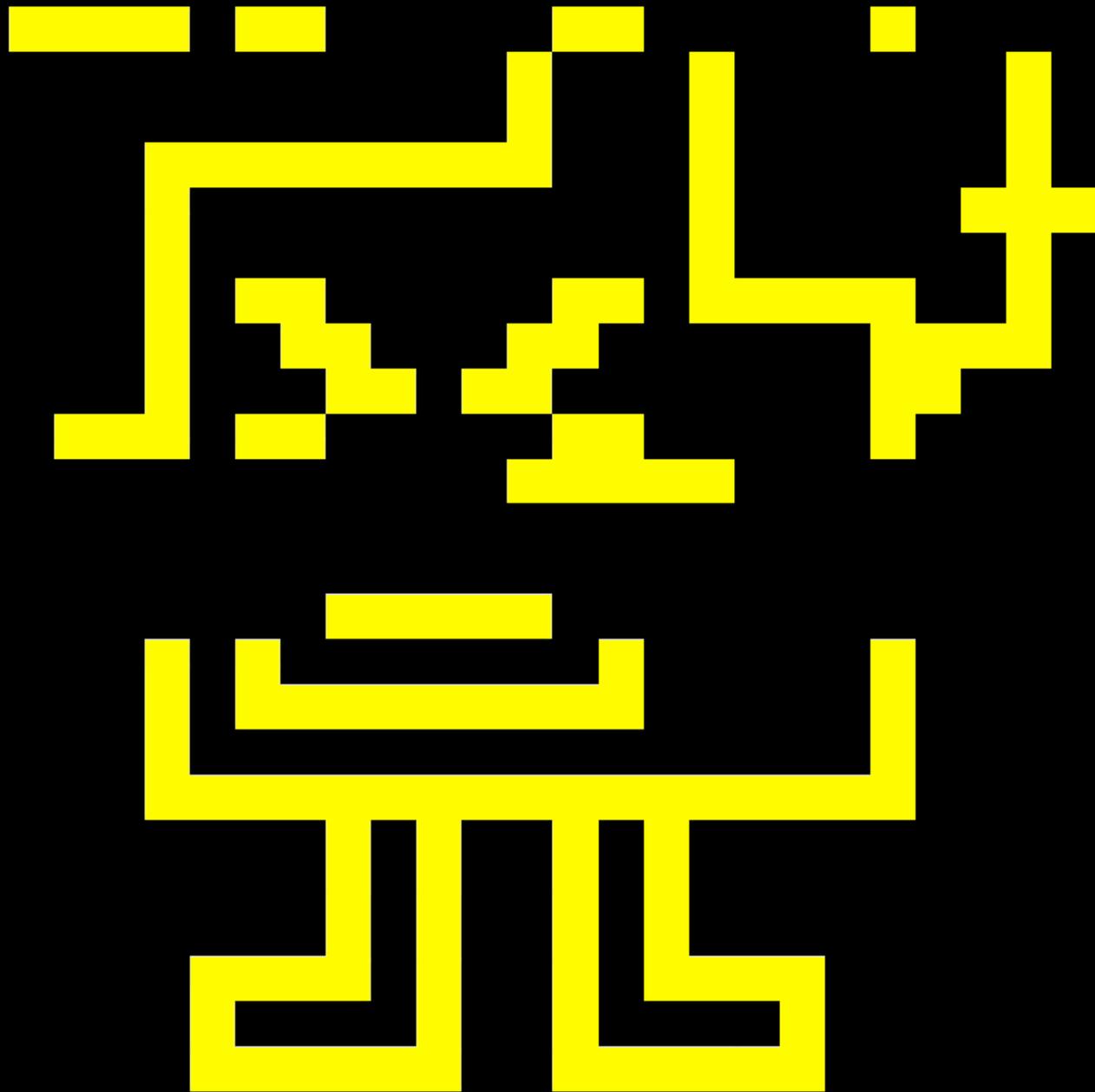
Warriors with their prey



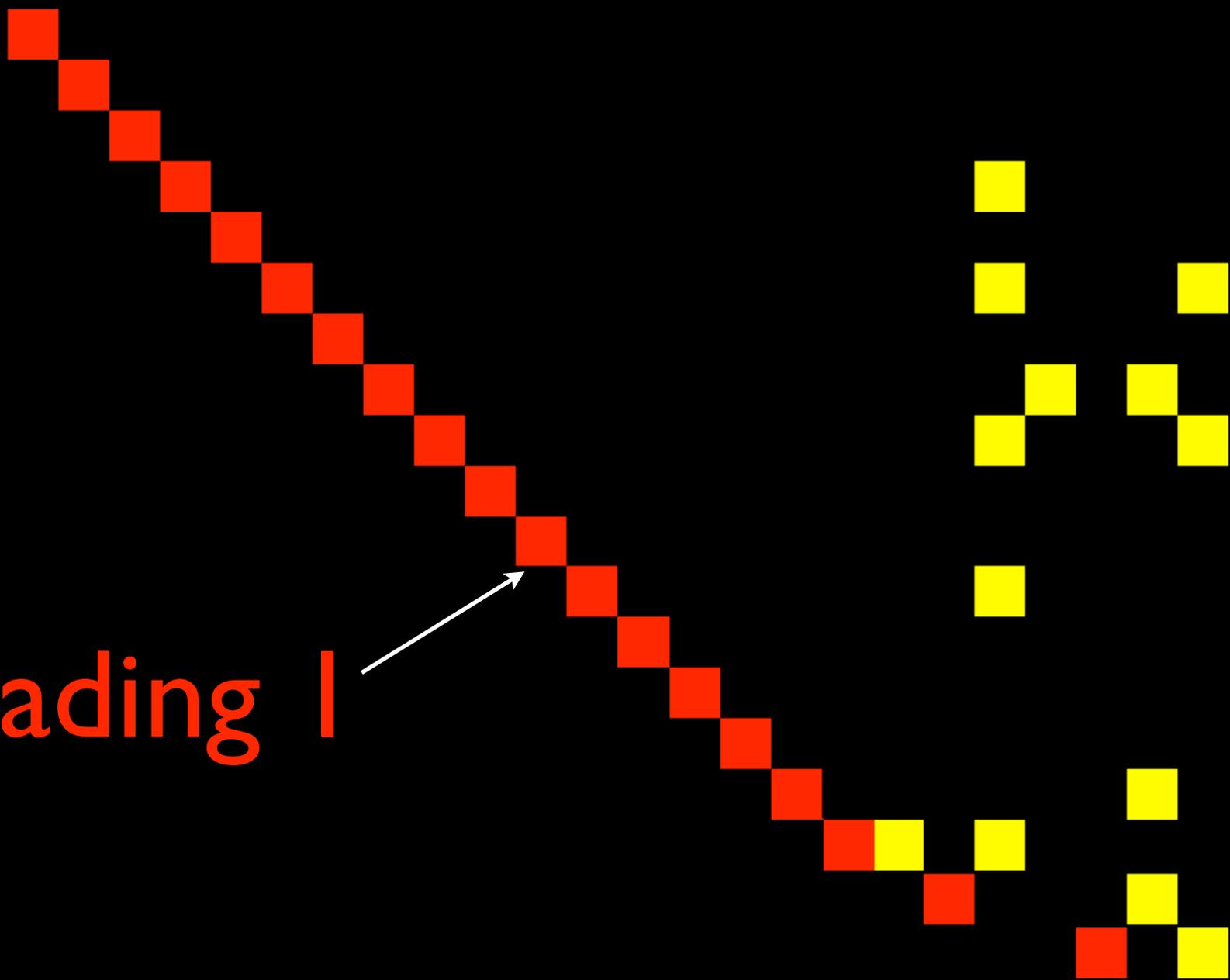
The milkshake as a matrix



The rref death of a milkshake



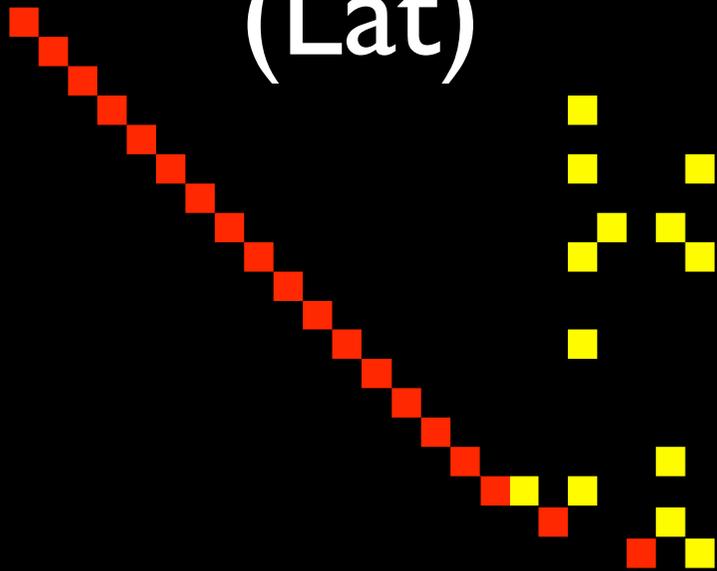
leading 1



There is a kernel!

kernel = nukleus

(Lat)



Does this mean, there was a nuclear device
in that milkshake?

Row reduced echelon form

1. Every first nonzero element in a row is 1
2. Leading columns otherwise only contain 0's
3. Every row above leading row leads to the left

“Leaders like to be first, do not like other leaders in the same column and like leaders above them to be to their left.”

Row reduced echelon form?

1	4	2	0	0
0	0	0	1	0
0	0	0	0	0
0	0	0	0	1

Row reduced echelon form?

1	4	2	6	2
0	1	2	3	4
0	0	0	0	0
0	0	0	0	0

Row reduced echelon form?

1	5	0	0	0	1
0	0	1	0	0	3
0	0	0	1	0	4
0	0	0	0	1	5
0	0	0	0	0	0

Inverting a matrix

Second
blackboard
problem ✓

invert the following matrix

1	0	3
1	1	4
3	0	10

1

0

3

1

1

4

3

0

10



1	0	3
---	---	---

1	1	4
---	---	---

3	0	10
---	---	----

1	0	0
---	---	---

0	1	0
---	---	---

0	0	1
---	---	---



1	0	3
---	---	---

1	1	4
---	---	---

3	0	10
---	---	----

1	0	0
---	---	---

0	1	0
---	---	---

0	0	1
---	---	---

1	0	0
---	---	---

0	1	0
---	---	---

0	0	1
---	---	---

10	0	-3
----	---	----

2	1	-1
---	---	----

-3	0	1
----	---	---



Which 2x2 matrices are their own inverse?

4
examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

are there
more?

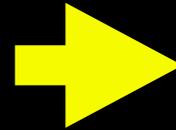
There are more!

Hint:

change basis

Matrix algebra

$$A X = B$$



$$X = A^{-1} B$$

$$(A X)^{-1} + B = C$$



$$X = A^{-1} (C - B)^{-1}$$

Multiply from the left or the right

II. Linear transformations

T plays well with 0,
addition and scalar
multiplication:

$$T(0) = 0$$

$$T(x+y) = T(x) + T(y)$$

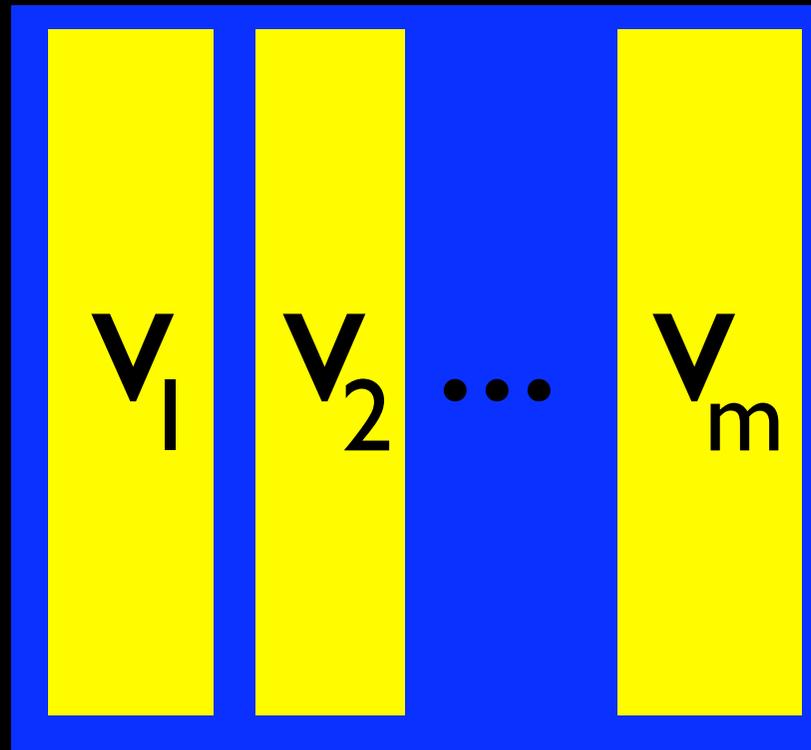
$$T(\lambda x) = \lambda T(x)$$

How do we express T
as a matrix?

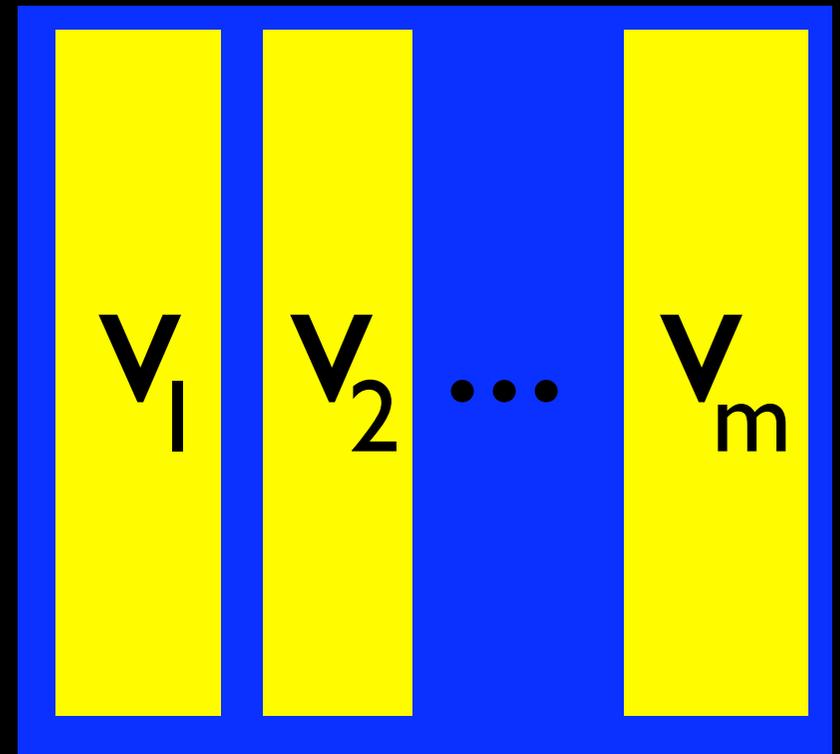
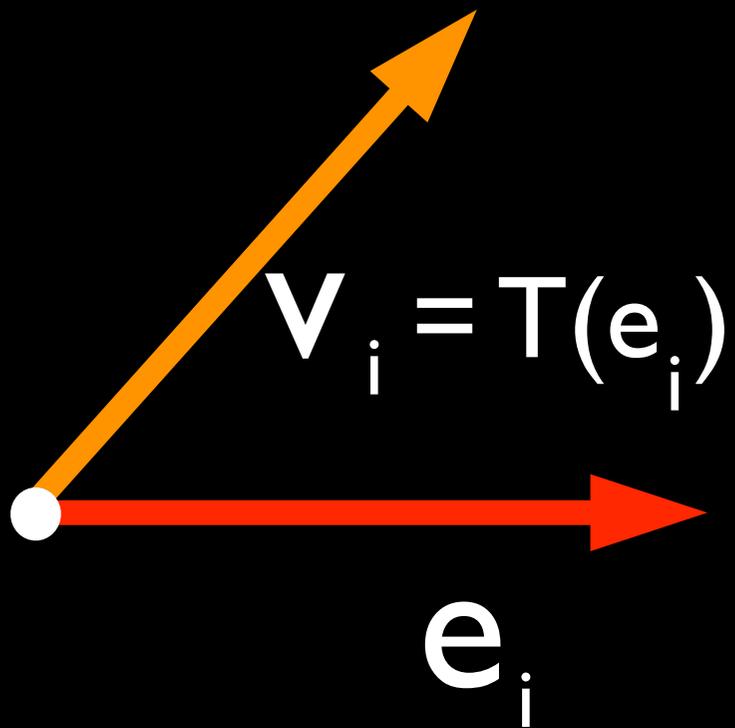


Key Fact:

The columns of A are the images of the basis vectors.



Geometry \longleftrightarrow Algebra



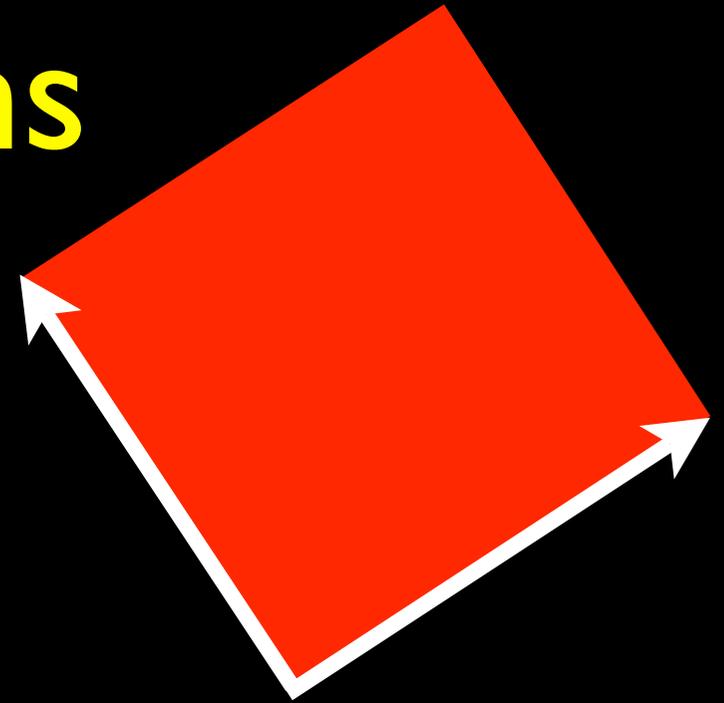
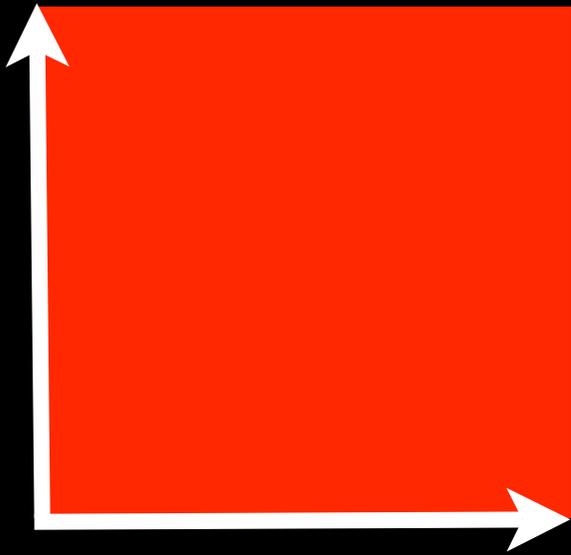
Example

What does the following transformation do?

1	0	0
0	1	2
0	0	1

Examples of transformations

rotations

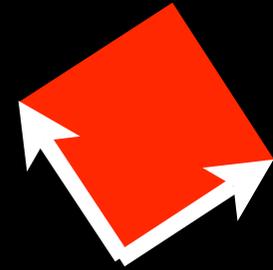
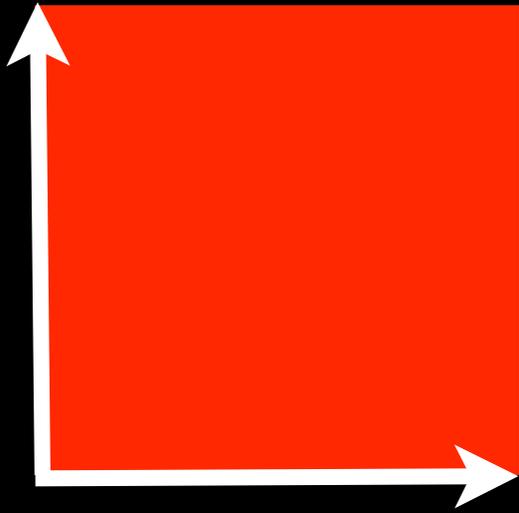


$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$





rotation-dilation

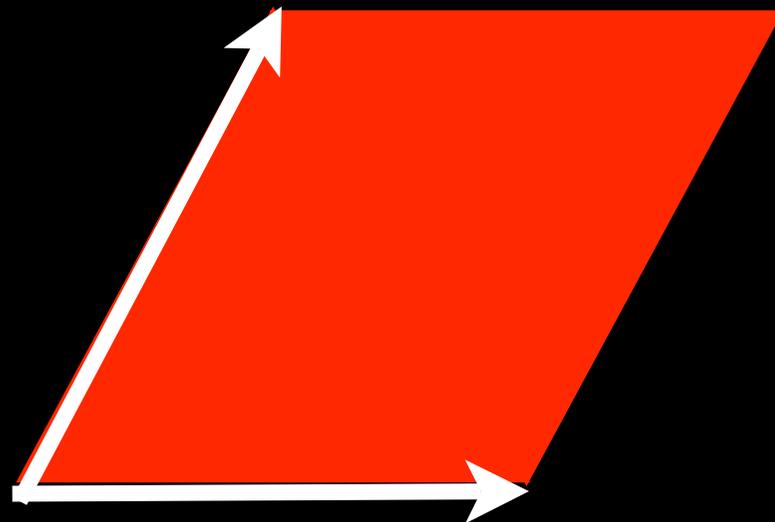
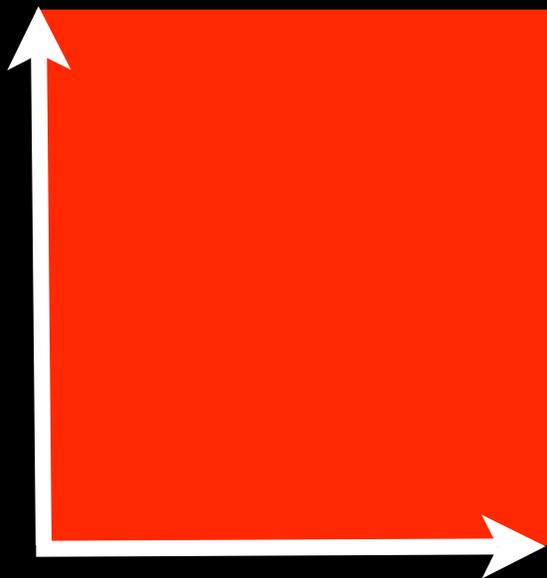


$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$





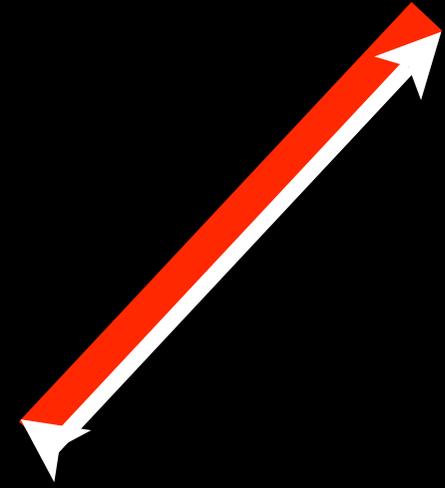
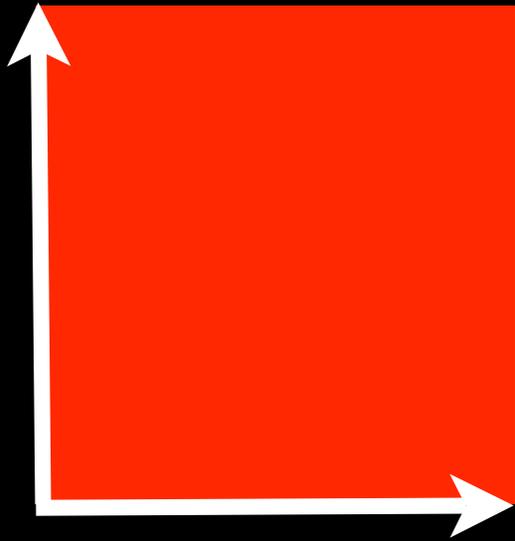
shears



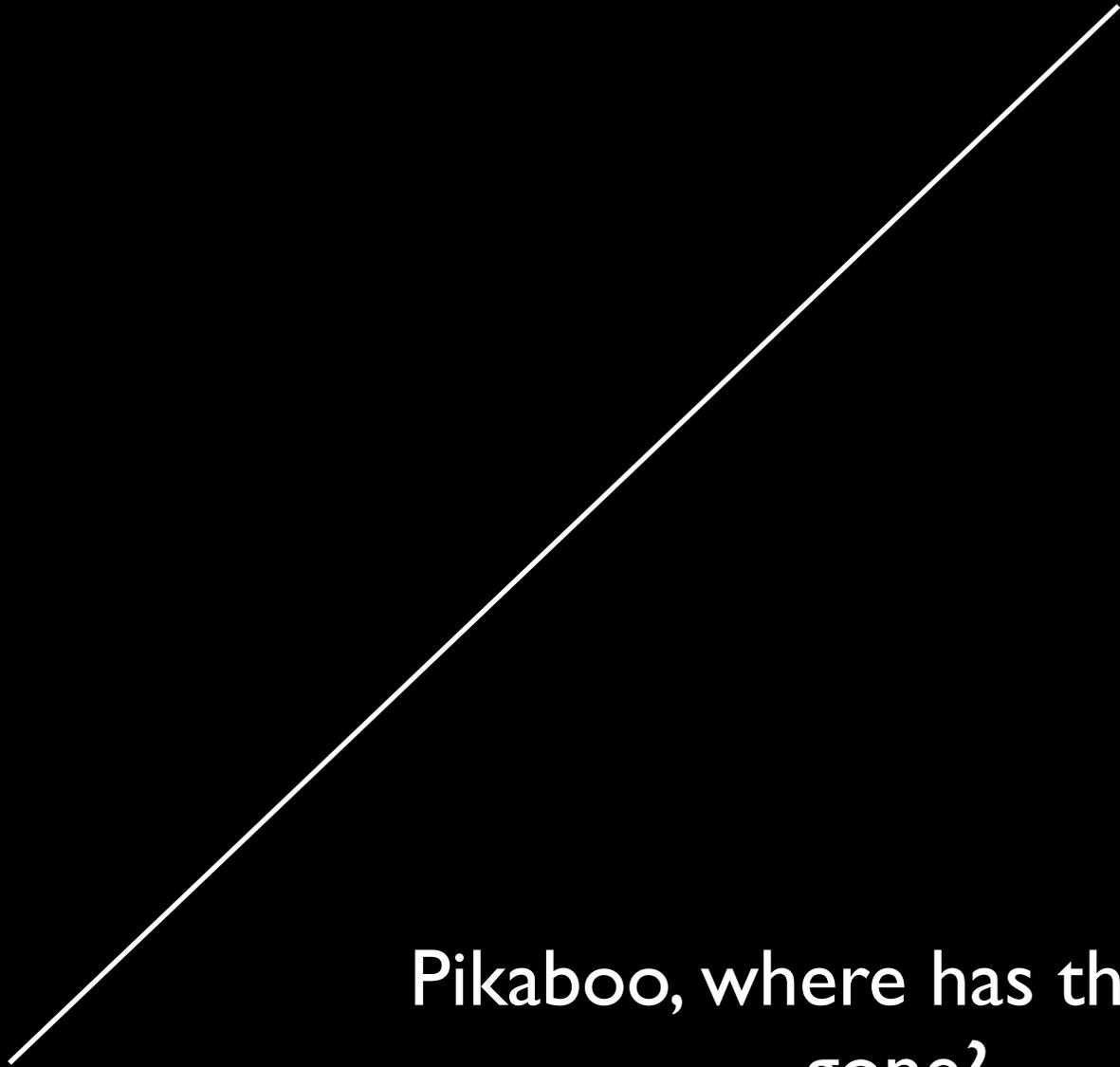
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



projections

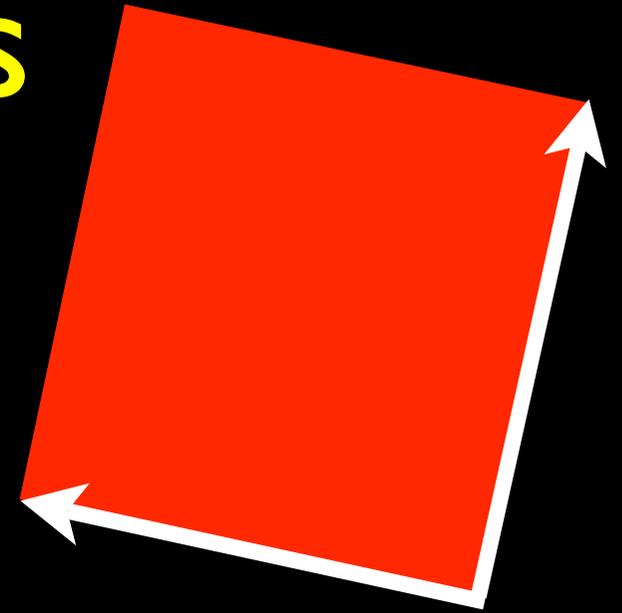
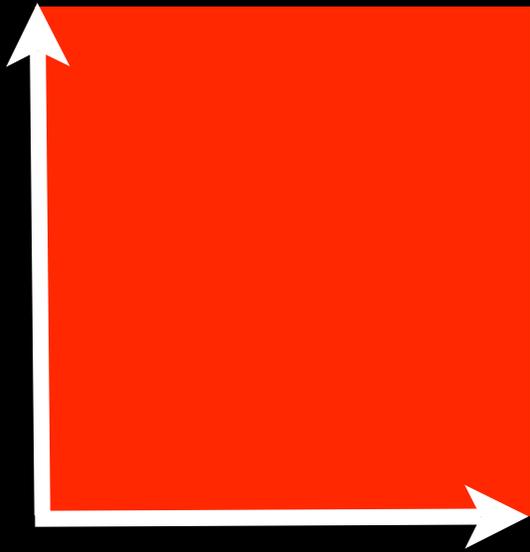


$$\begin{bmatrix} u_1 u_1 & u_2 u_1 \\ u_1 u_2 & u_2 u_2 \end{bmatrix}$$



Pikaboo, where has the queen
gone?

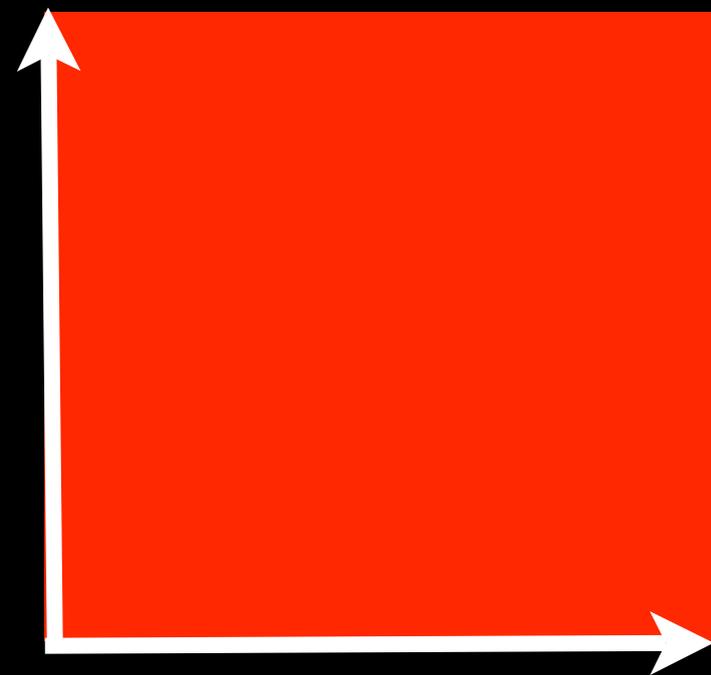
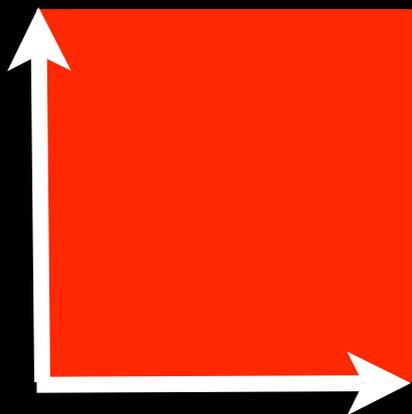
reflections



$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$



dilations



$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$





Third
blackboard
problem ✓

Find the matrix of the
transformation in \mathbb{R}^4

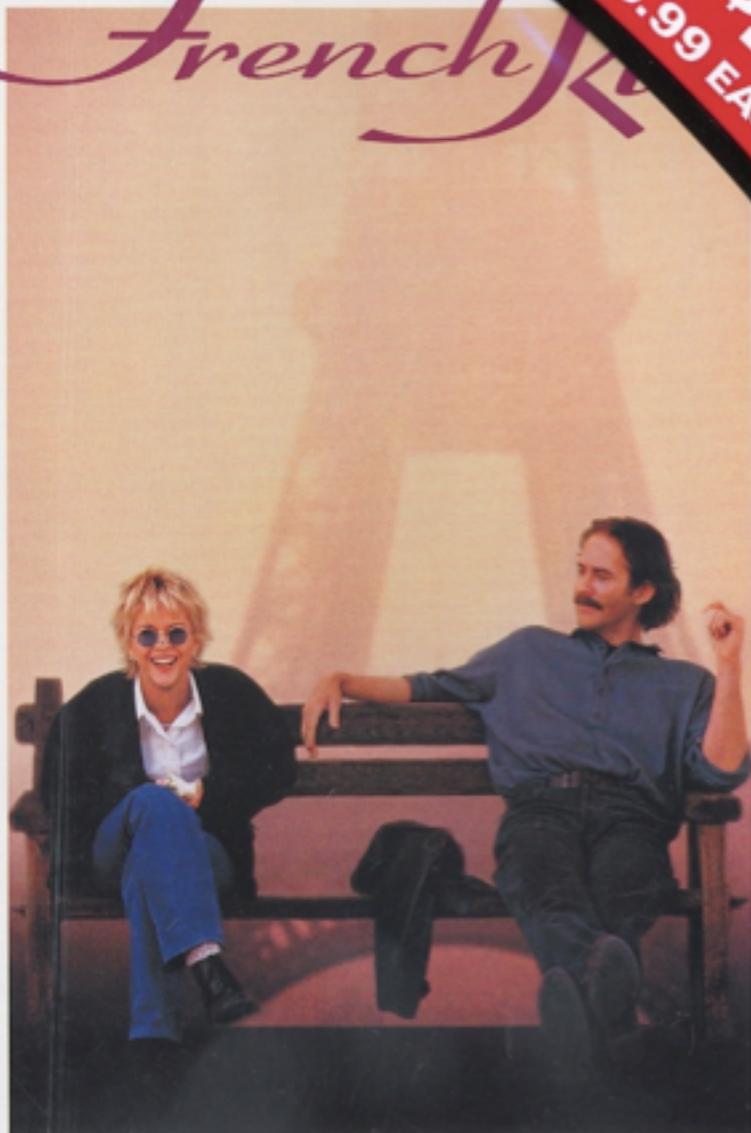
which reflects at the
 xz plane then reflects
at the yz plane.

Quiz coming up!



MEG RYAN

French Kiss



3 FOR \$25
OR \$9.99 EACH

"DELIGHTFUL! RYAN AND KLINE ARE IRRESISTIBLE."

BY RITA KEMPLEY, THE WASHINGTON POST

DVD
VIDEO

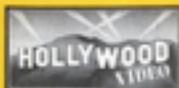
MEG RYAN | RUSSELL CROFT

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E. FOR HONOR. FOR MANKIND.

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**NEW
MOVIE**
\$9.99

STEVEN SODERBERGH'S FILM

**THE
LAST
MAY**

DVD



What is the length of

$$10^{100}$$

6	8	0
8	-6	0
0	0	1

0
1
0

The answer is

One googolplex

$10^{10^{100}}$

The name gogool was a term coined by Milton Sirotta (1929-1980), nephew of Edward Kasner (1878-1955). The googolplex number can not be written down in our universe (10^{80} atoms)

III. Basis

Linear independence

if

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_n \mathbf{v}_n = \mathbf{0}$$

$$\text{then, } a_1 = a_2 = \dots = a_n = 0$$

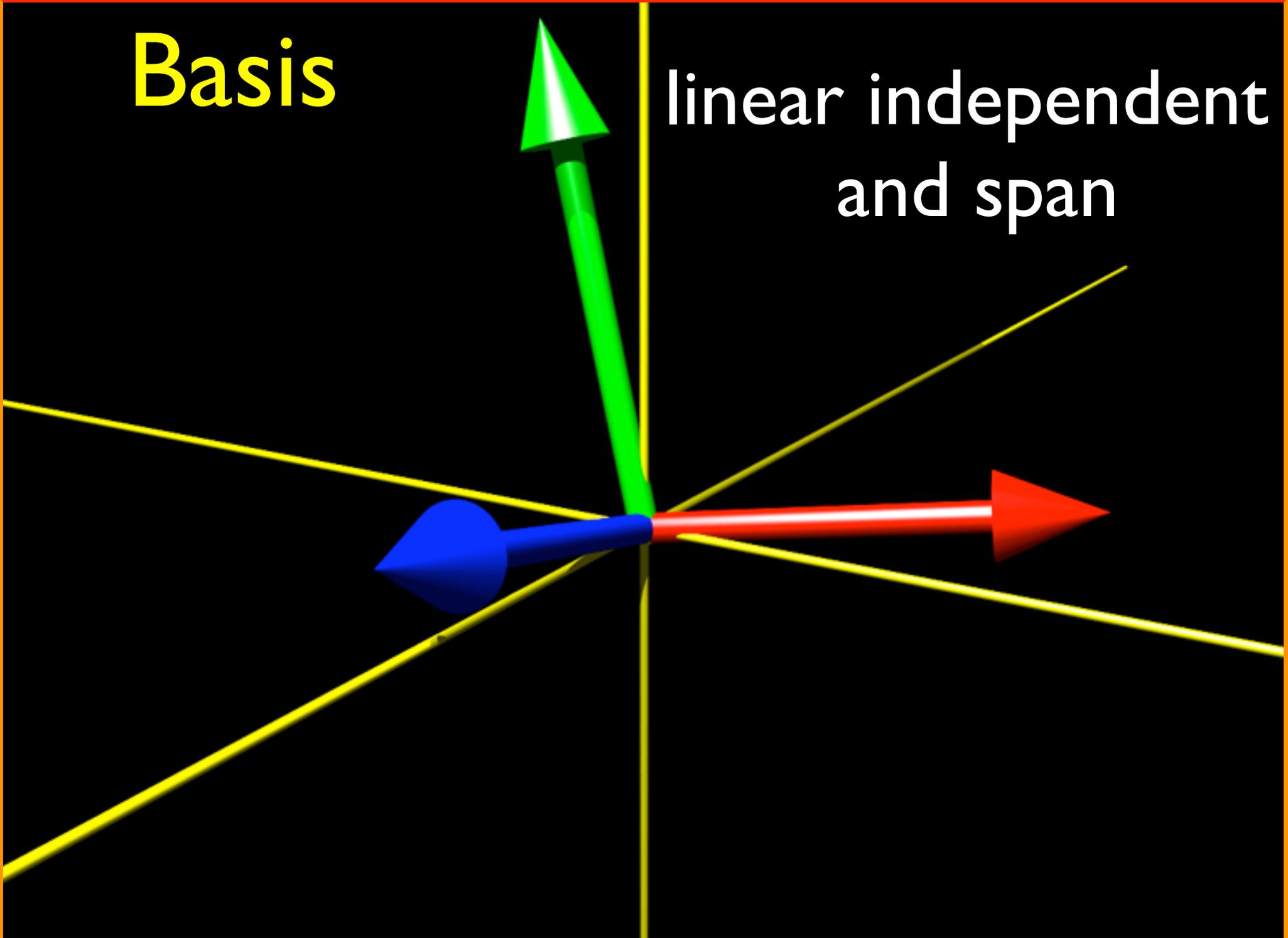
Spanning

every v in V can be written as

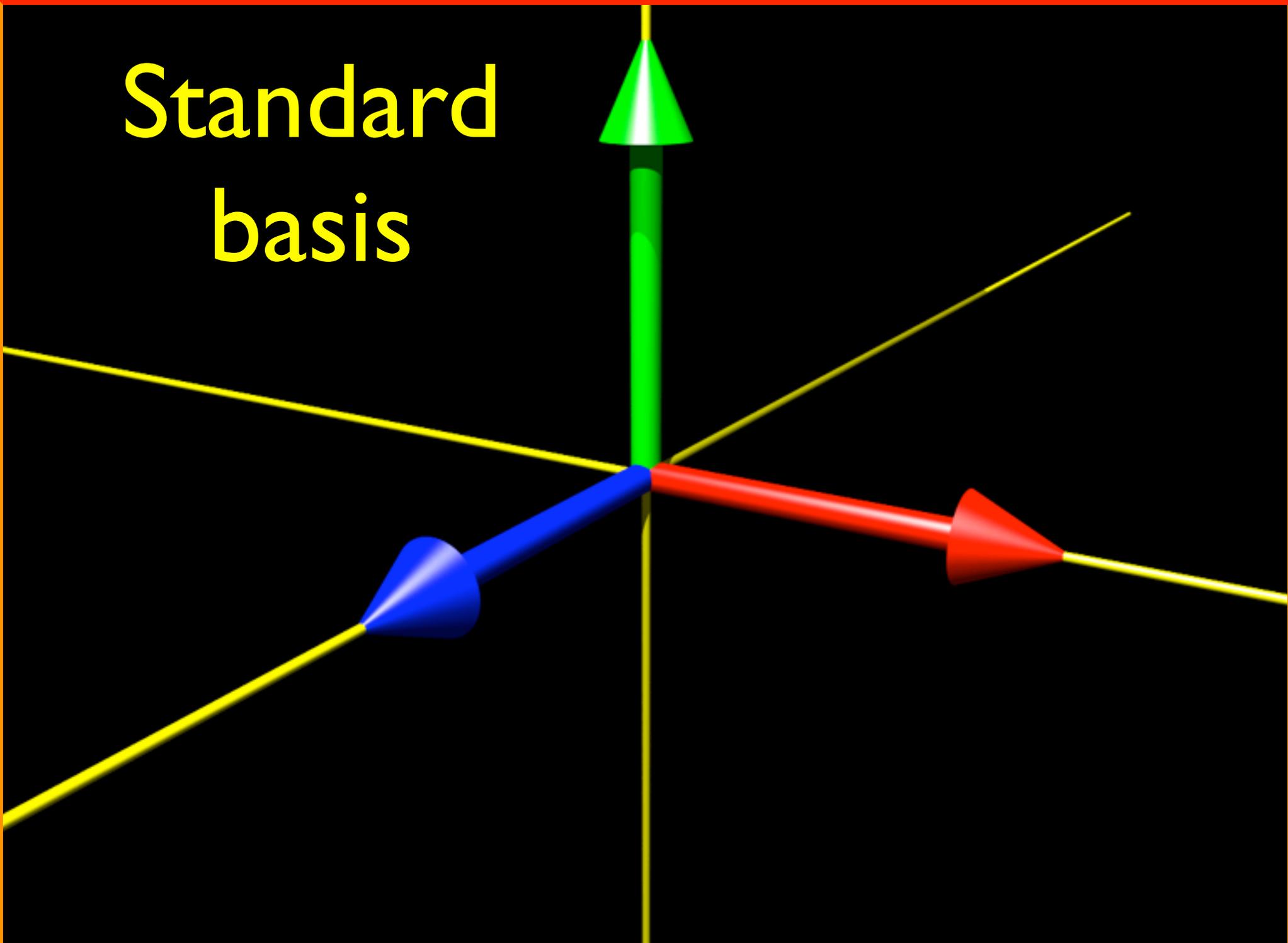
$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = v$$

Basis

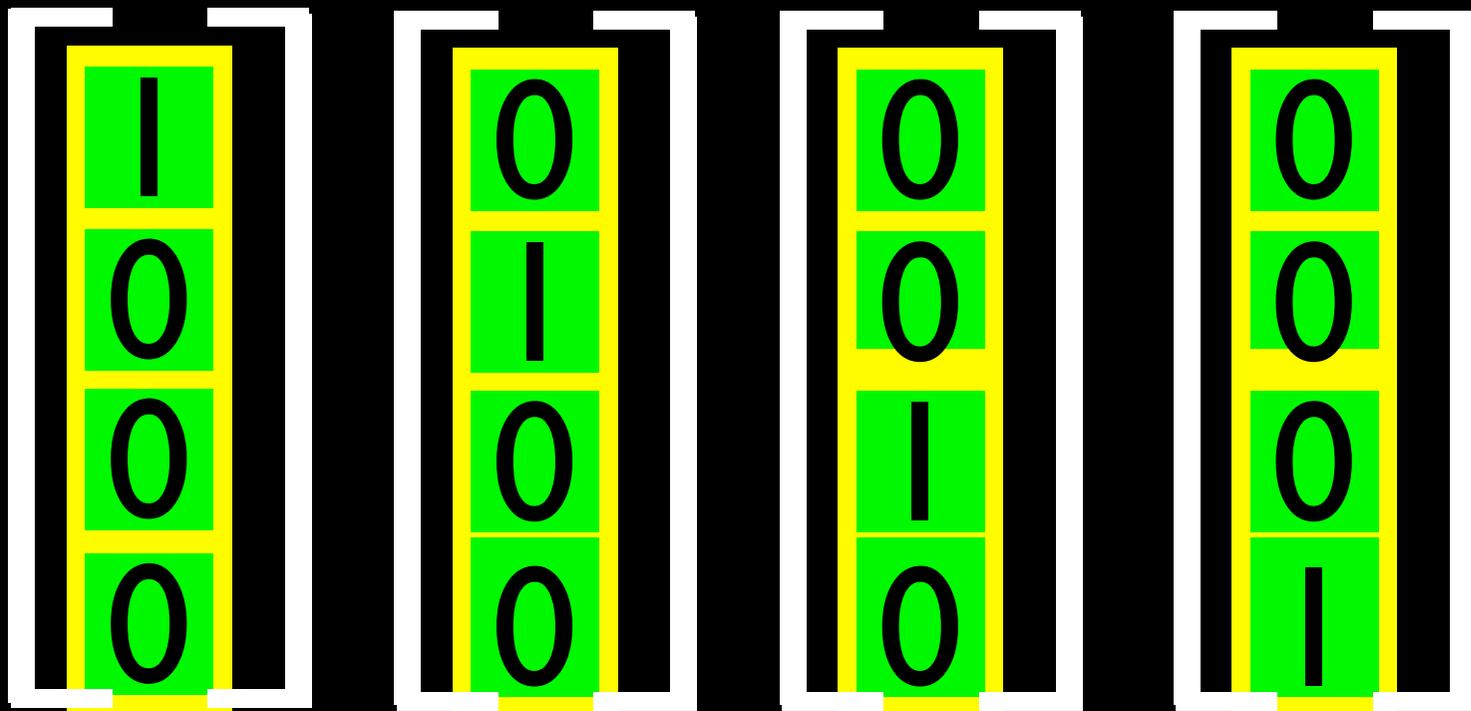
linear independent
and span



Standard
basis



Standard basis



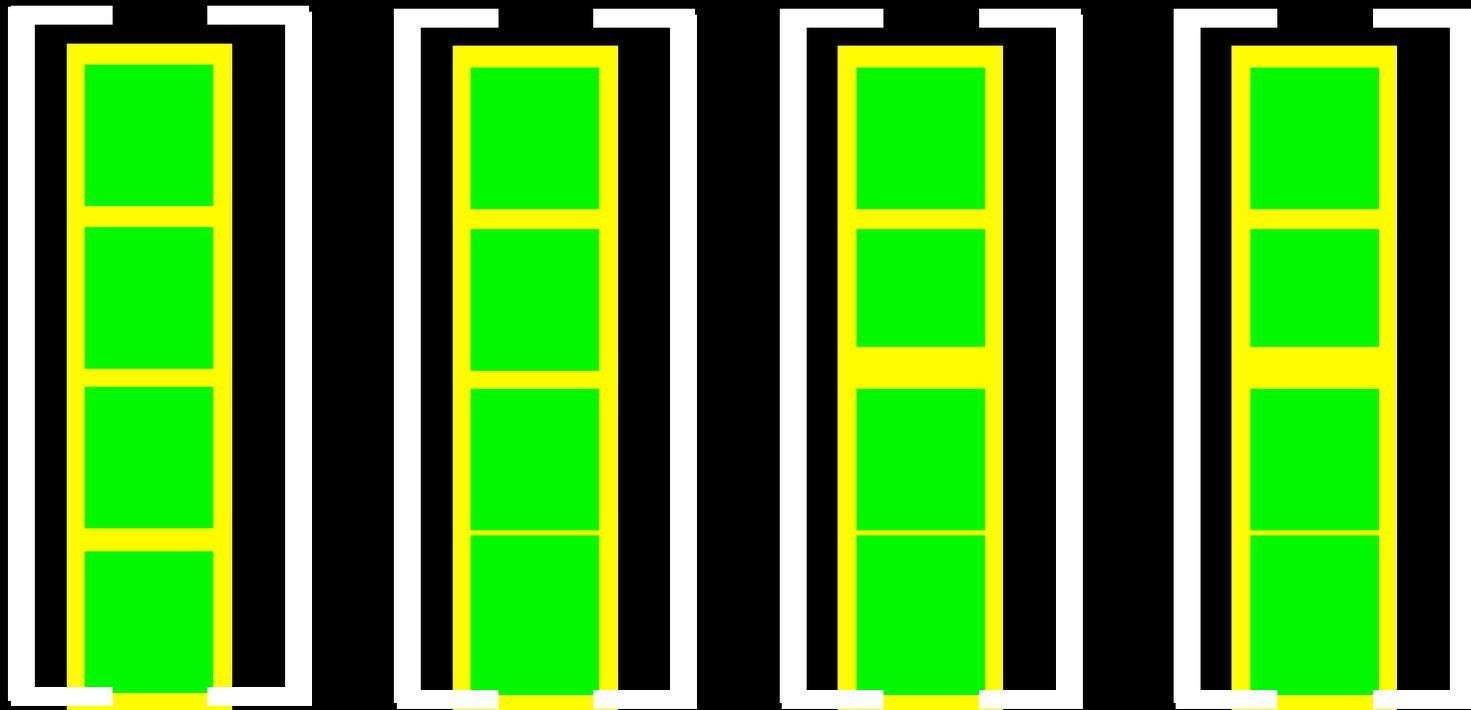
e_1

e_2

e_3

e_4

How do we check to have a basis?



v_1

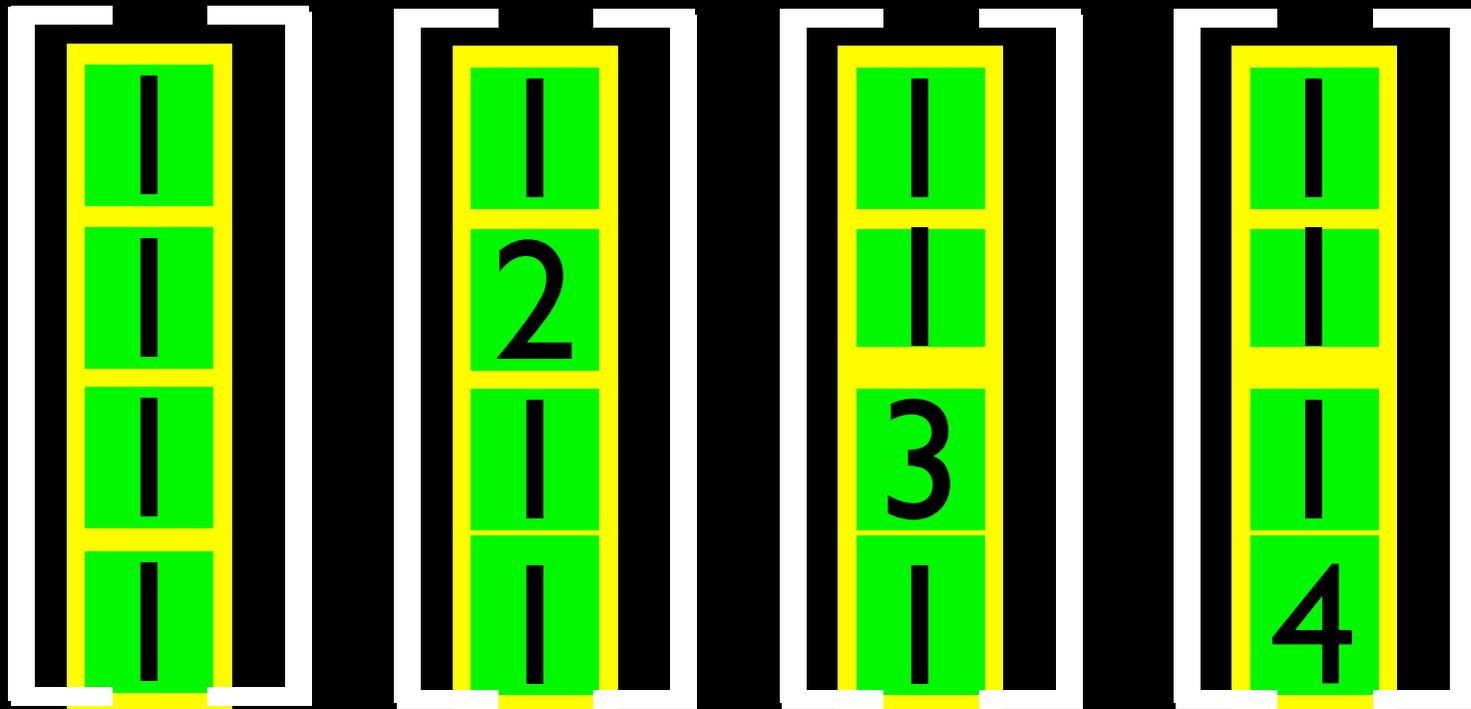
v_2

v_3

v_4

Fourth
blackboard
problem ✓

Is this a basis?



v_1

v_2

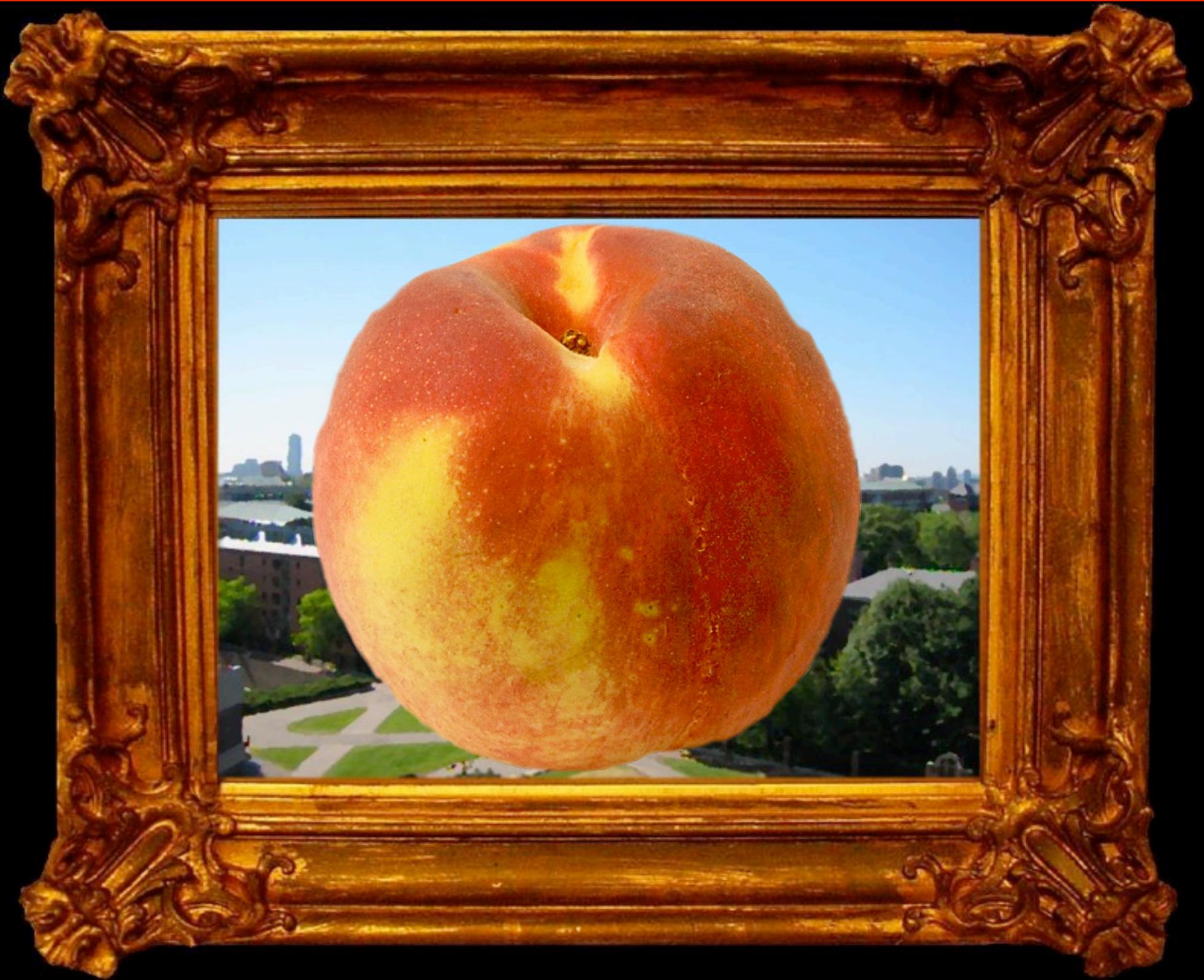
v_3

v_4

IV. Image and Kernel









Rene Magritte,
The son of man: 1963

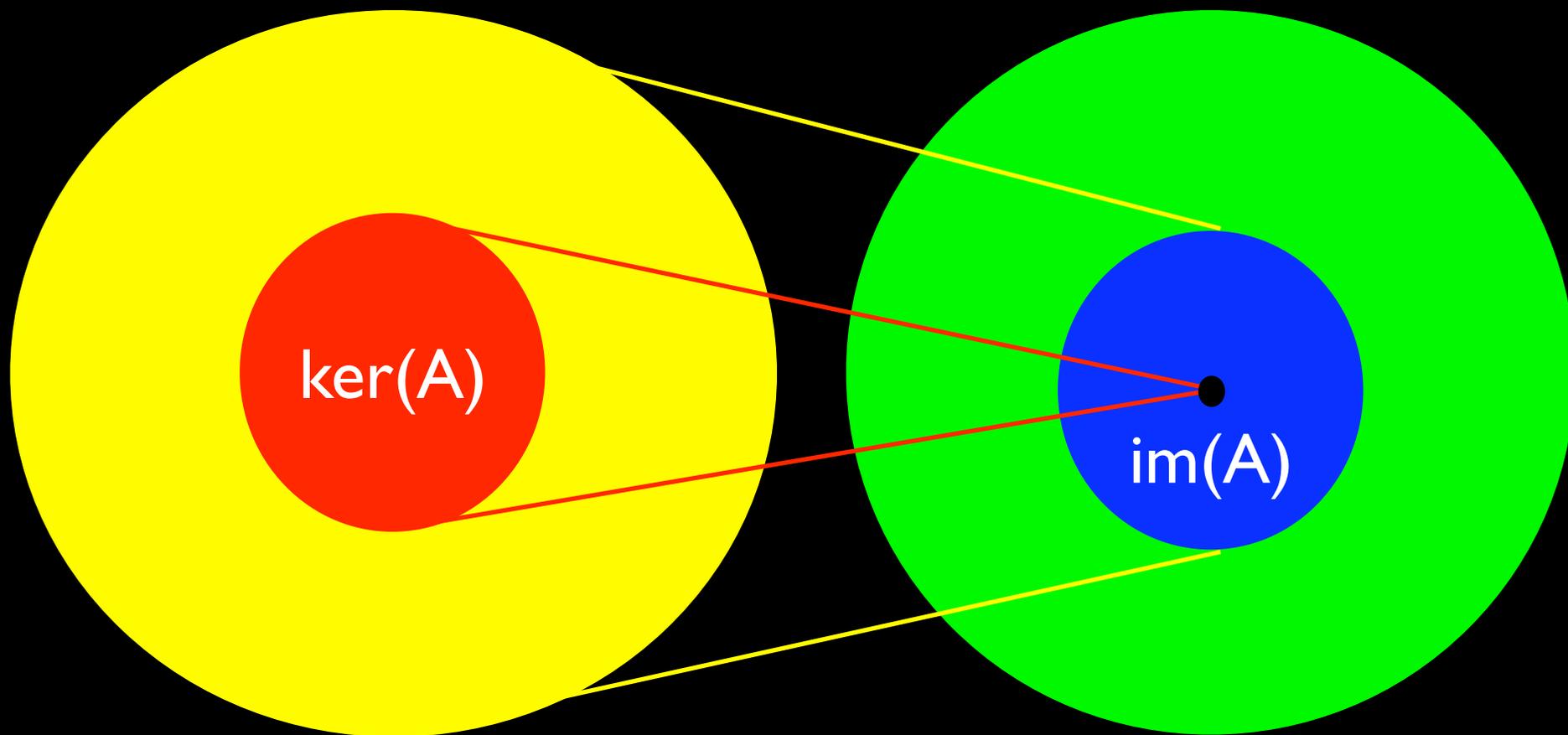


Oliver Knill
Son of a peach, 2007



Oliver Knill
Son of a peach, 2007

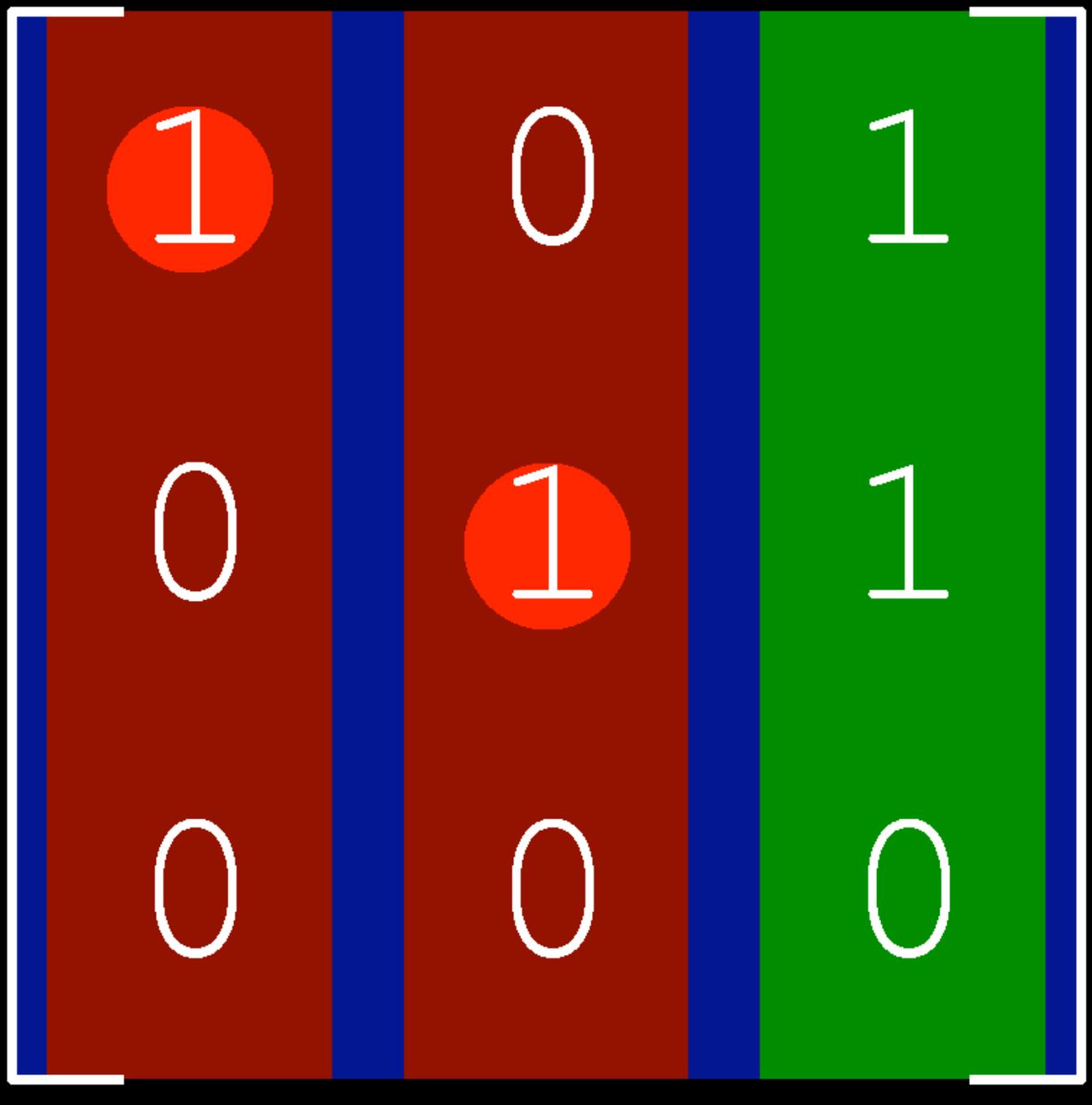
$$\text{im}(A) = \{ Ax \mid x \text{ in } V \}$$

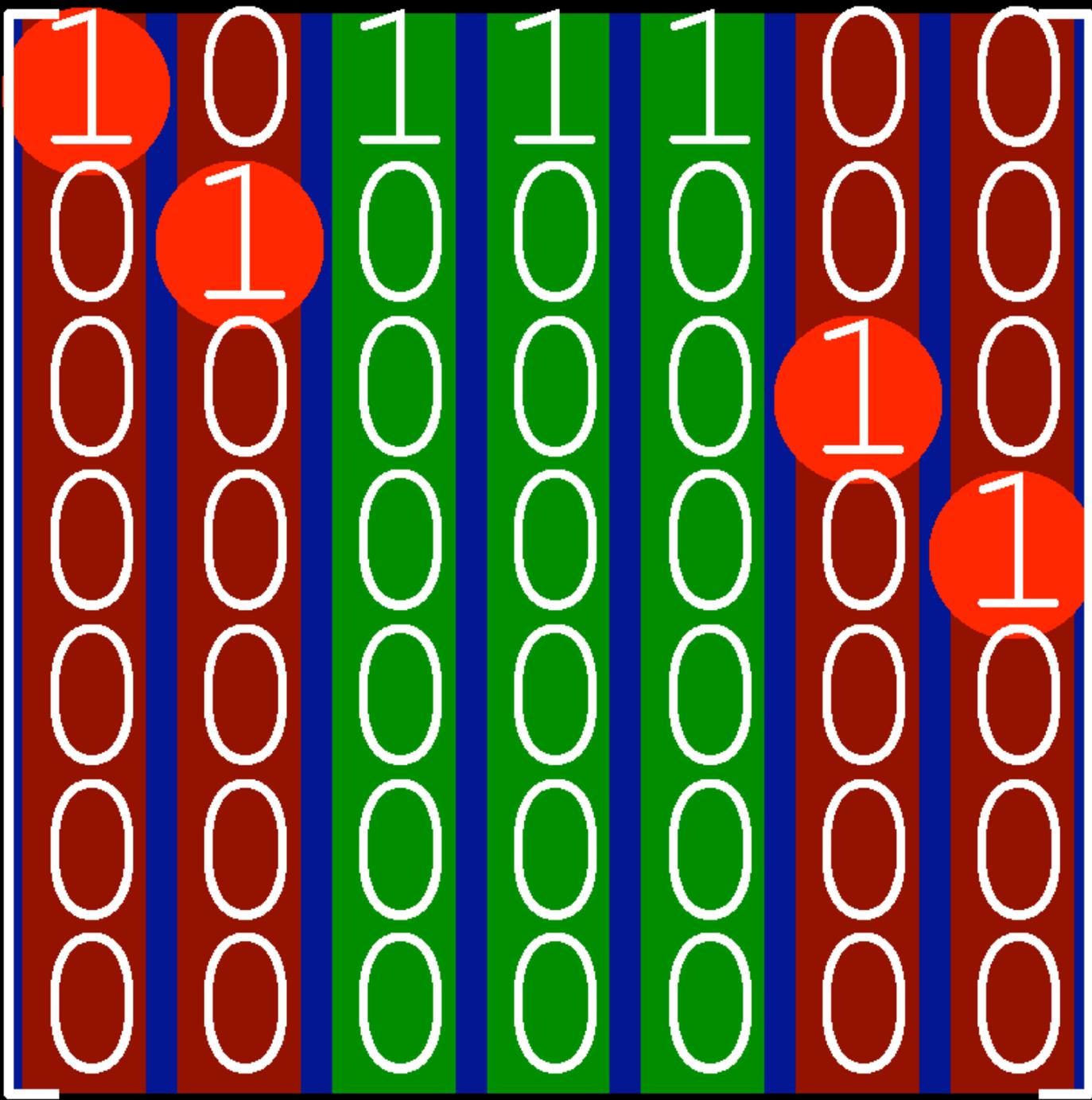


$$\text{ker}(A) = \{ x \mid Ax = 0 \}$$

How do we compute
a basis for the image
and kernel?

row reduce!





Dimension formula

rank nulley

$$\dim(\text{im}(A)) + \dim(\text{ker}(A)) = m$$

fundamental theorem of linear algebra.

rank nulley theorem

Pivot or
not pivot,
that is
here the
question

m

n

Computing the image:

The basis is the set of pivot columns.

Computing the kernel:

The basis is obtained by solving the linear system in row reduced echelon form and taking free variables.

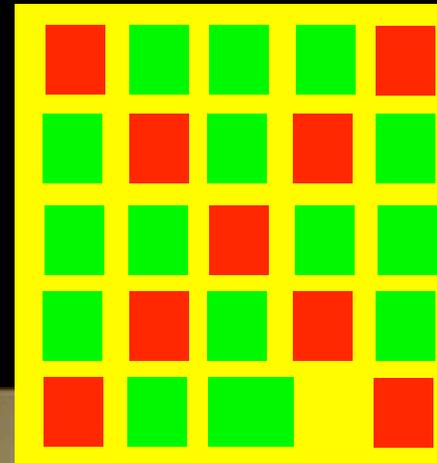
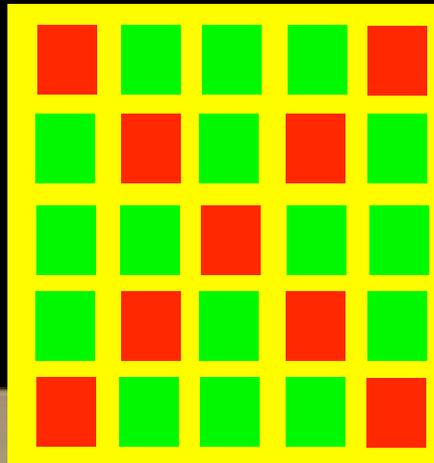
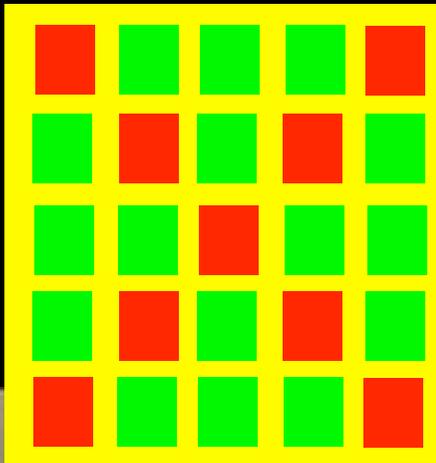
*Fifth
blackboard
problem*



Image and kernel of X matrix

The image shows a 5x5 matrix with a yellow background. The matrix is enclosed in large white brackets. The cells are colored red or green, and contain either a black '1' or a black '0'. The matrix is symmetric and has a block-like structure.

1	0	0	0	1
0	1	0	1	0
0	0	1	0	0
0	1	0	1	0
1	0	0	0	1



XXX (2003) , movie won Taurus award for this stunt.
18 cameras filmed in Auburn CA at 730 feet bridge.

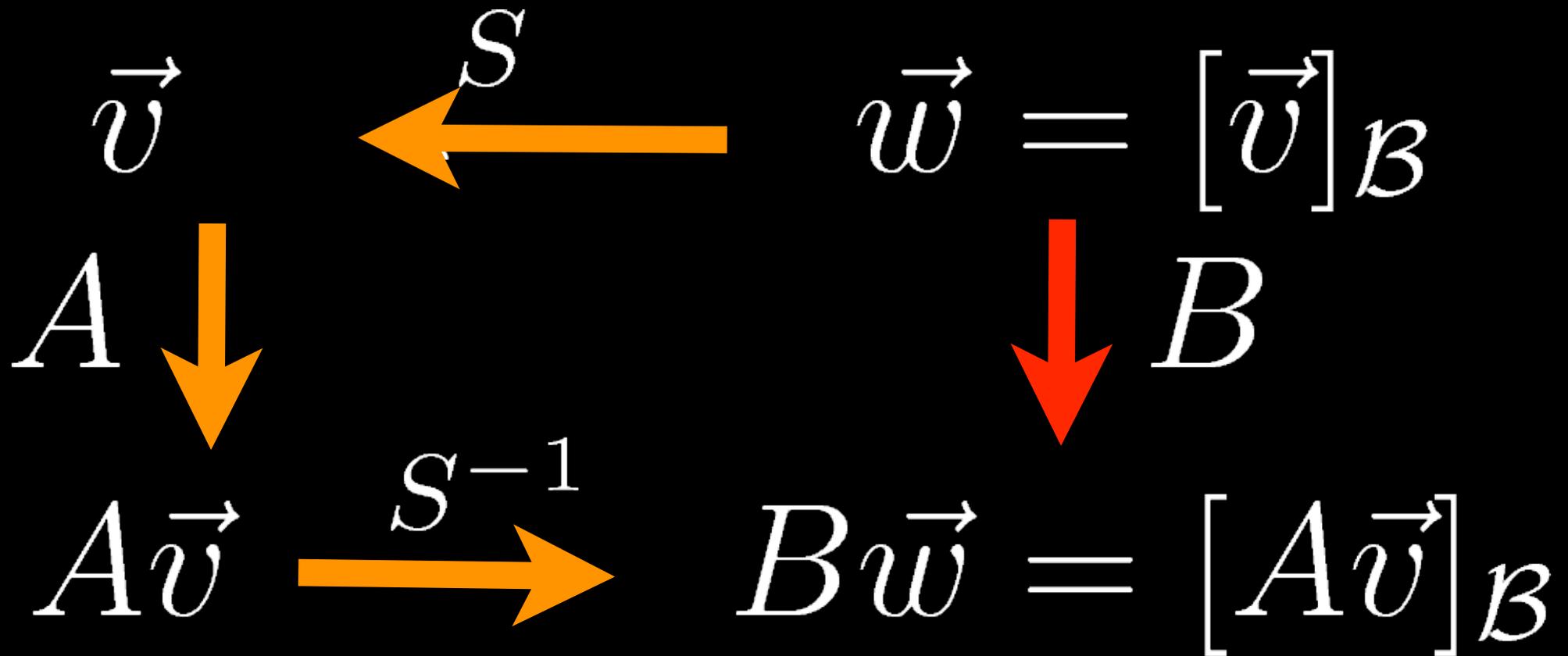
IV. Coordinates

v coordinates in standard basis

$$[v]_B = S^{-1} v$$

$[v]$ coordinates in basis B

$$\mathbf{B} = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$$



*Sixth
blackboard
problem*



Problem

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

What are the B
coordinates of $v =$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

S

I

S

I

1	0	0	0
1	0	0	0
1	1	1	0
1	1	1	1

1	-1	0	0
0	1	-1	0
0	0	1	-1
0	0	0	1

$$S^{-1}v =$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$=$$

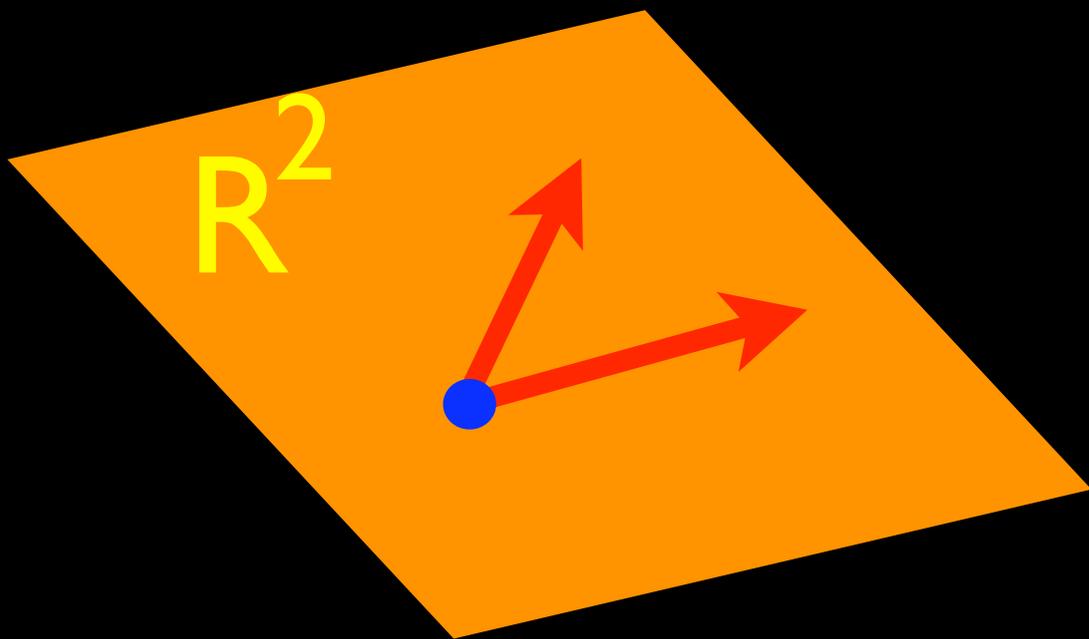
$$\begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

V. Linear Spaces

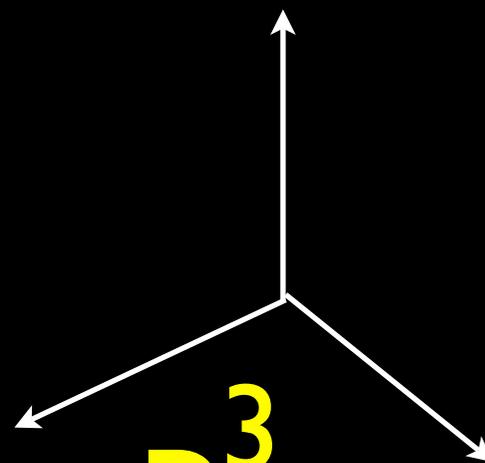
R^0



R^1



R^2



R^3

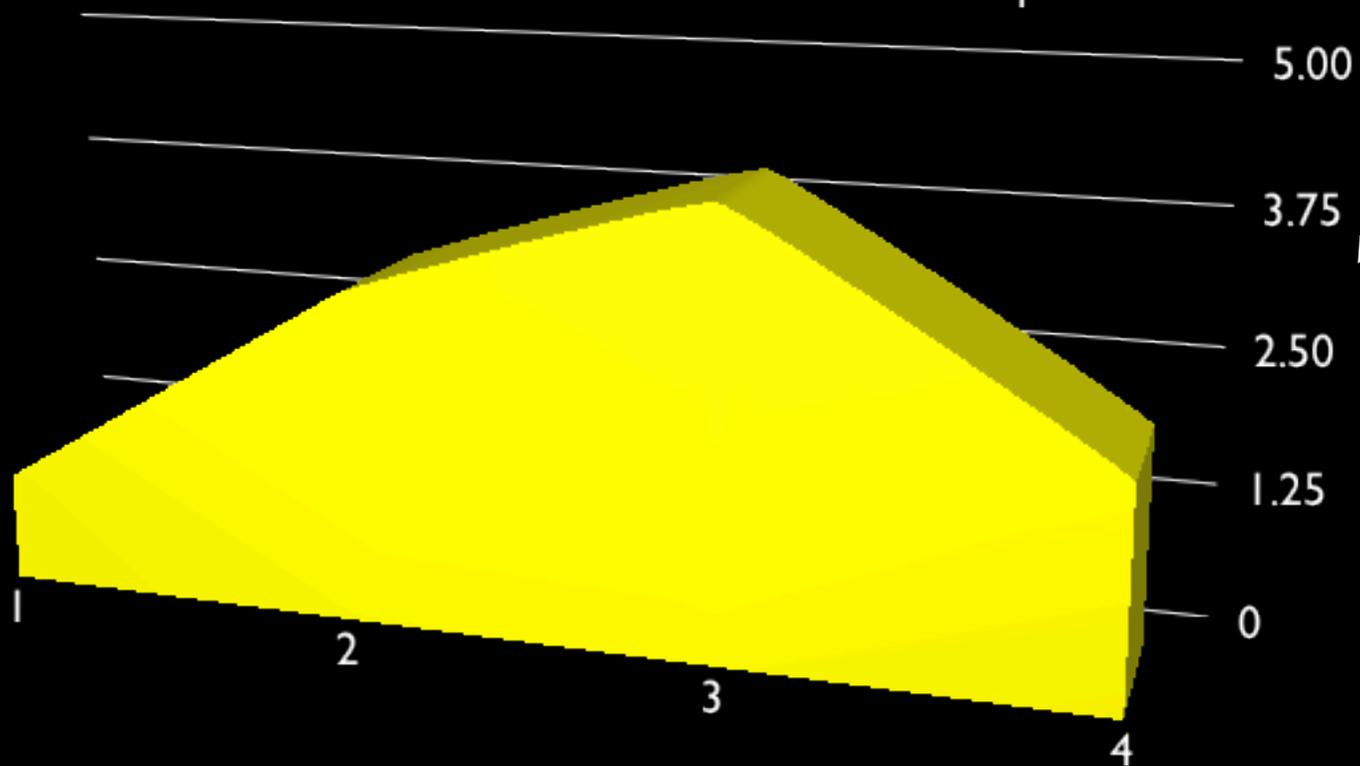
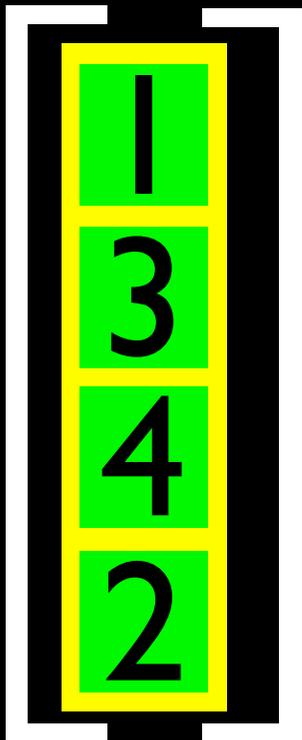
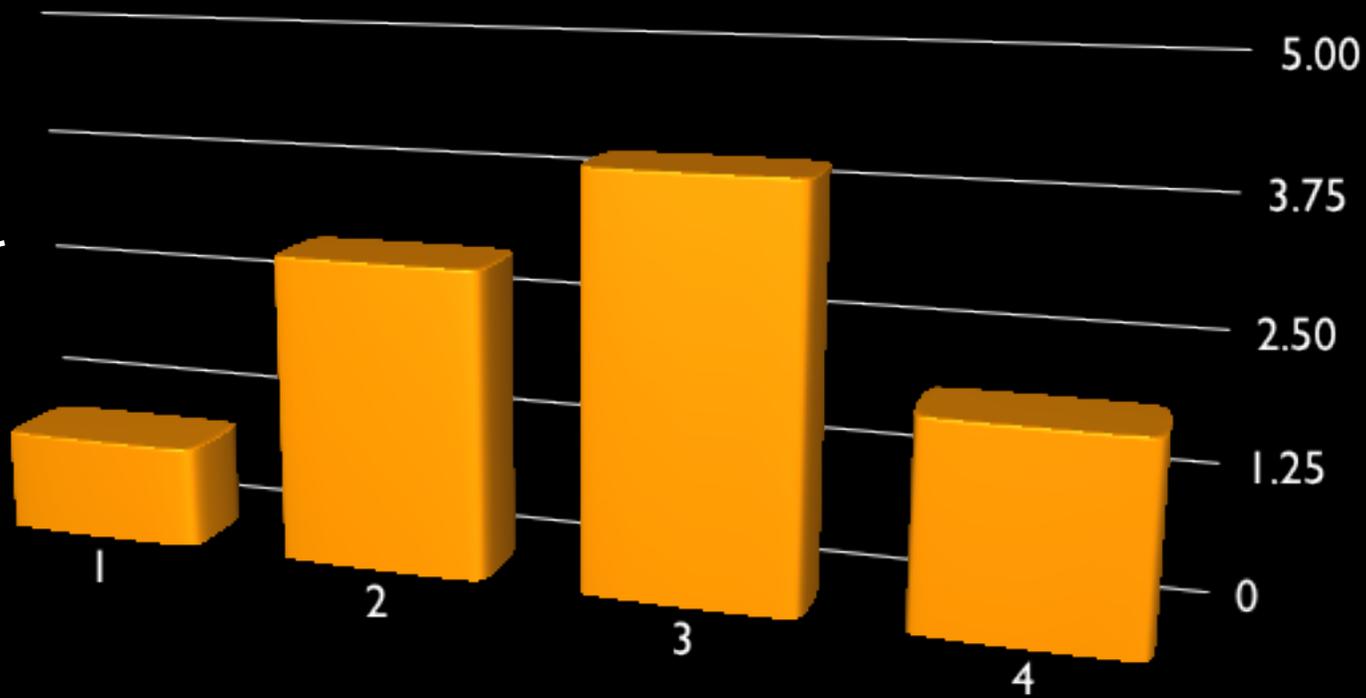
\mathbb{R}^n = space of all maps from $\{1,2,\dots,n\}$ to \mathbb{R}

we write the map

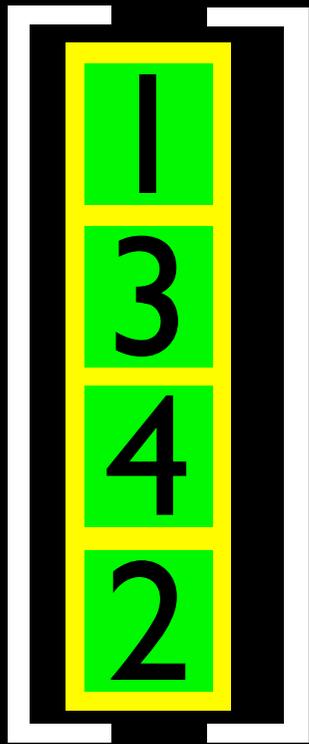
$$\{1,2,3\} \longrightarrow \{5,7,-8\}$$

as

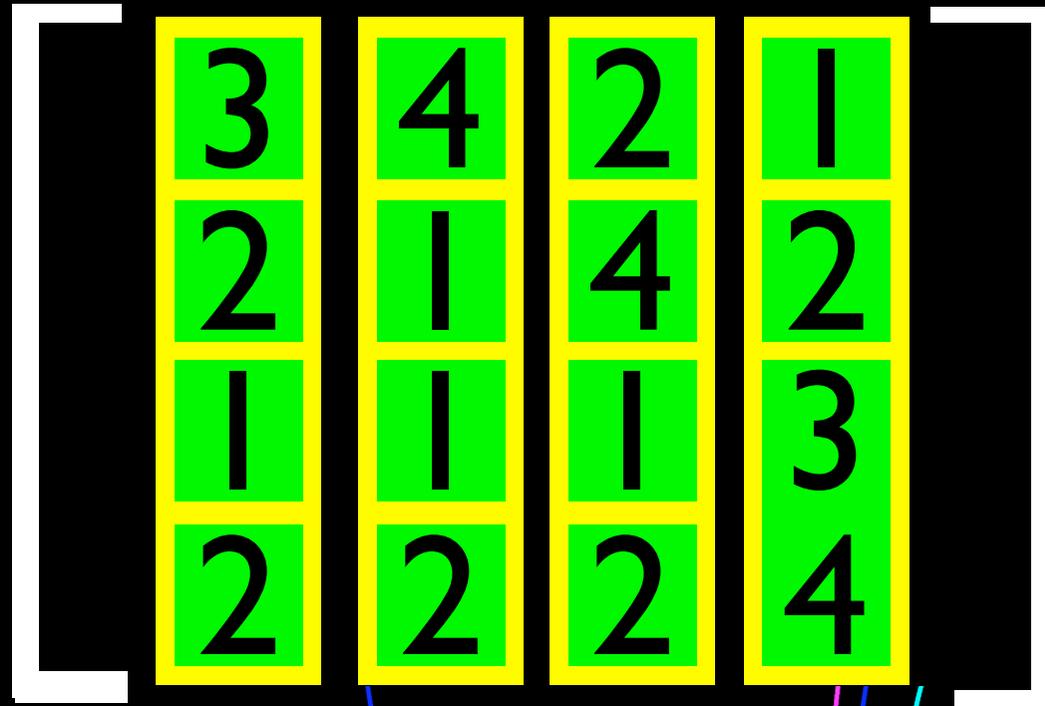
$$\begin{bmatrix} 5 \\ 7 \\ -8 \end{bmatrix}$$



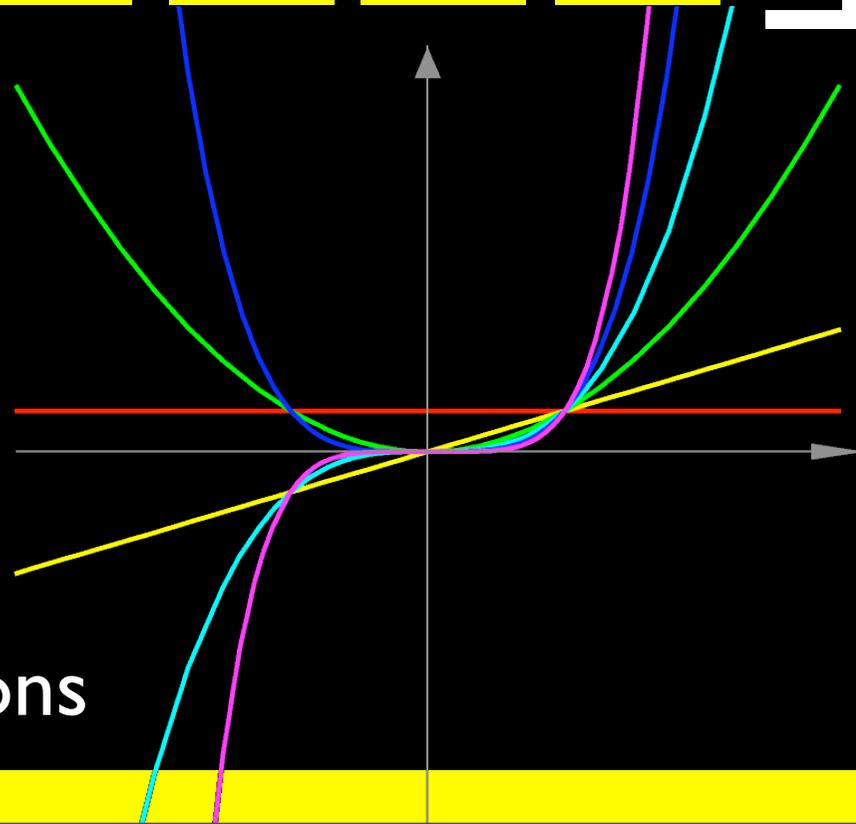
Type of linear spaces



vectors

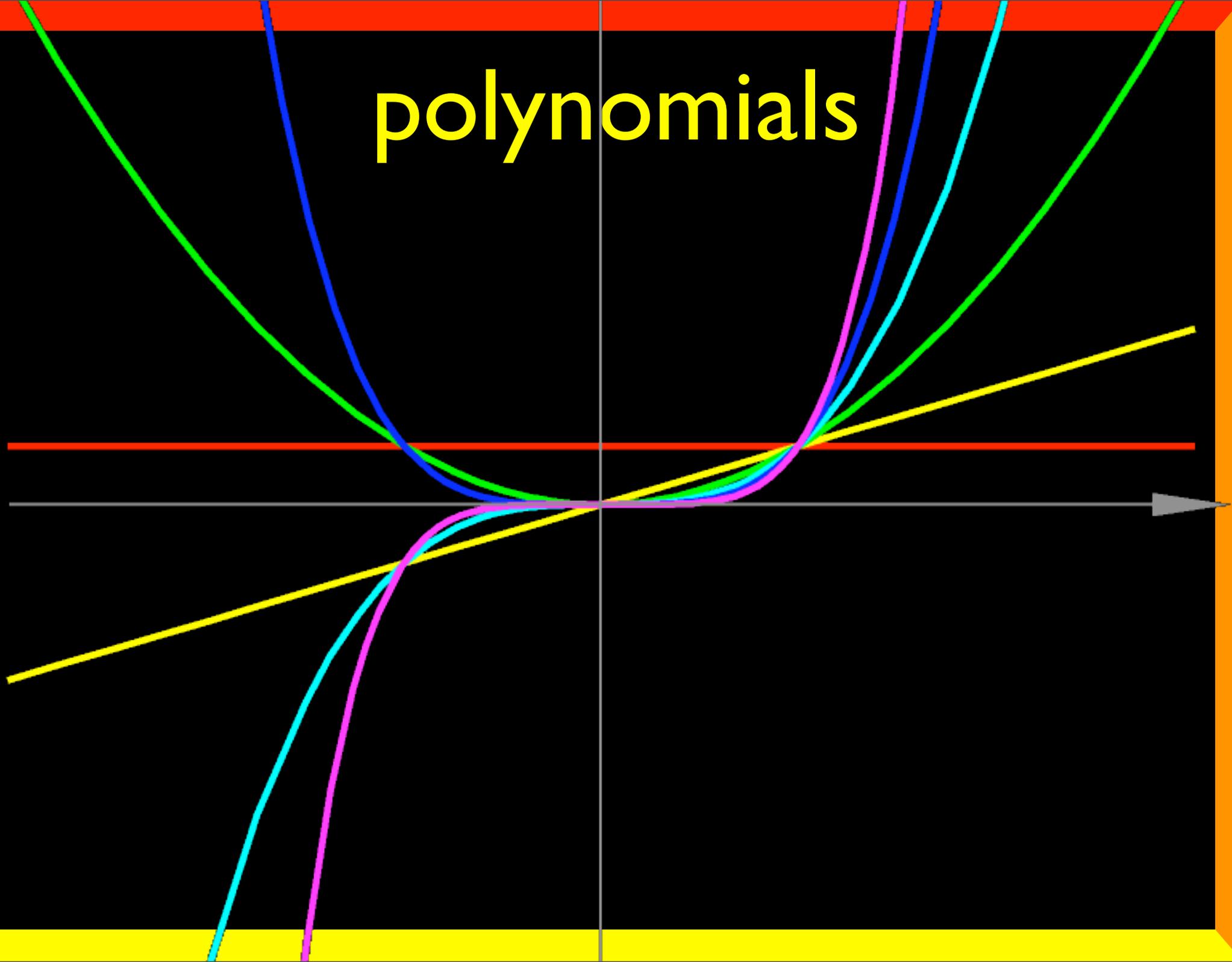


matrices



functions

polynomials



solutions to differential equations

Solutions to linear differential equations are linear spaces.

Example:

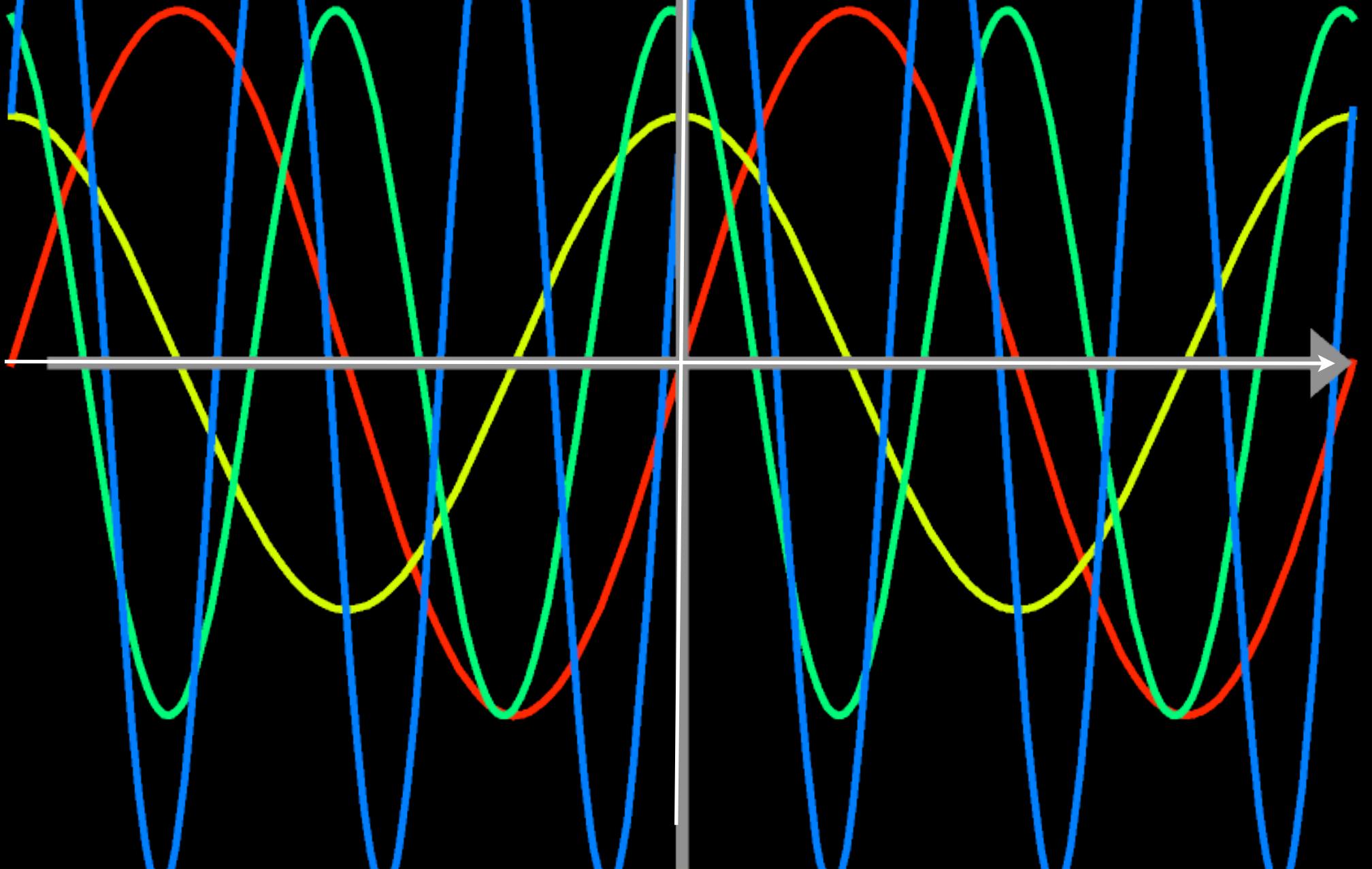
$$X = \{ f \text{ in } C^{\infty}(\mathbb{R}) \mid f''(x) = -f(x) \}$$

matrices

$$\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}$$

$$-3 \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

functions



Seventh
blackboard
problem ✓

Is this a linear space?

$$X = \{f \text{ in } P_2 \mid f''(0) = 1\}$$

$$X = \{f \text{ in } P_3 \mid f(0) = 1\}$$

Trap!



This is
not a
linear
space!

The end

