

**THE INVERSE**

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HOMEWORK: 2.3: 12,30,44,40\*,42\*, 2.4: 14,40,28\*,72\*

INVERTIBLE TRANSFORMATIONS. A map  $T$  from  $X$  to  $Y$  is **invertible** if there is for every  $y \in Y$  a **unique** point  $x \in X$  such that  $T(x) = y$ .



EXAMPLES.

- 1)  $T(x) = x^3$  is invertible from  $X = \mathbf{R}$  to  $X = Y$ .
- 2)  $T(x) = x^2$  is not invertible from  $X = \mathbf{R}$  to  $X = Y$ .
- 3)  $T(x, y) = (x^2 + 3x - y, x)$  is invertible from  $X = \mathbf{R}^2$  to  $Y = \mathbf{R}^2$ .
- 4)  $T(\vec{x}) = Ax$  linear and  $\text{rref}(A)$  has an empty row, then  $T$  is not invertible.
- 5) If  $T(\vec{x}) = Ax$  is linear and  $\text{rref}(A) = 1_n$ , then  $T$  is invertible.

INVERSE OF LINEAR TRANSFORMATION. If  $A$  is a  $n \times n$  matrix and  $T: \vec{x} \mapsto Ax$  has an inverse  $S$ , then  $S$  is linear. The matrix  $A^{-1}$  belonging to  $S = T^{-1}$  is called the **inverse matrix** of  $A$ .

First proof: check that  $S$  is linear using the characterization  $S(\vec{a} + \vec{b}) = S(\vec{a}) + S(\vec{b}), S(\lambda\vec{a}) = \lambda S(\vec{a})$  of linearity. Second proof: construct the inverse using Gauss-Jordan elimination.

FINDING THE INVERSE. Let  $1_n$  be the  $n \times n$  identity matrix. Start with  $[A|1_n]$  and perform Gauss-Jordan elimination. Then

$$\text{ref}([A|1_n]) = [1_n|A^{-1}]$$

Proof. The elimination process solves  $A\vec{x} = \vec{e}_i$  simultaneously. This leads to solutions  $\vec{v}_i$  which are the columns of the inverse matrix  $A^{-1}$  because  $A^{-1}\vec{e}_i = \vec{v}_i$ .

EXAMPLE. Find the inverse of  $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ .

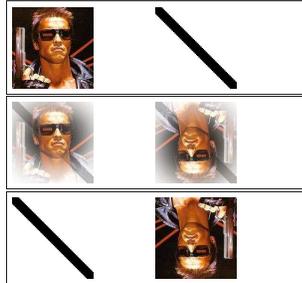
$$\left[ \begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [A \mid 1_2]$$

$$\left[ \begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [\dots \mid \dots]$$

$$\left[ \begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [\dots \mid \dots]$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [1_2 \mid A^{-1}]$$

The inverse is  $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$ .



THE INVERSE OF LINEAR MAPS  $R^2 \mapsto R^2$ :

If  $ad - bc \neq 0$ , the inverse of a linear transformation  $\vec{x} \mapsto Ax$  with  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is given by the matrix  $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$ .

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

DIAGONAL:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

ROTATION:

$$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(-\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

ROTATION-DILATION:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}, r^2 = a^2 + b^2$$

BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

NONINVERTIBLE EXAMPLE. The projection  $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is a non-invertible transformation.

MORE ON SHEARS. The shears  $T(x, y) = (x + ay, y)$  or  $T(x, y) = (x, y + ax)$  in  $\mathbf{R}^2$  can be generalized. A shear is a linear transformation which fixes some line  $L$  through the origin and which has the property that  $T(\vec{x}) - \vec{x}$  is parallel to  $L$  for all  $\vec{x}$ . Shears are invertible.

PROBLEM.  $T(x, y) = (3x/2 + y/2, y/2 - x/2)$  is a shear along a line  $L$ . Find  $L$ .

SOLUTION. Solve the system  $T(x, y) = (x, y)$ . You find that the vector  $(1, -1)$  is preserved.

MORE ON PROJECTIONS. A linear map  $T$  with the property that  $T(T(x)) = T(x)$  is a projection. Examples:  $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$  is a projection onto a line spanned by a unit vector  $\vec{y}$ .

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections where the high frequencies are cut away.

MORE ON ROTATIONS. A linear map  $T$  which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form  $x \mapsto A(x)$  with  $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ .

An example of a rotations in three dimensions are  $\vec{x} \mapsto Ax$ , with  $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . It is a rotation around the  $z$  axis.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples:

**2D reflections at the origin:**  $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ , **2D reflections at a line**  $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$ .

**3D reflections at origin:**  $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . **3D reflections at a line**  $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . By

the way: in any dimensions, to a reflection at the line containing the unit vector  $\vec{u}$  belongs the matrix  $[A]_{ij} = 2(u_i u_j) - [1_n]_{ij}$ , because  $[B]_{ij} = u_i u_j$  is the matrix belonging to the projection onto the line.

The reflection at a line containing the unit vector  $\vec{u} = [u_1, u_2, u_3]$  is  $A = \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 - 1 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 - 1 \end{bmatrix}$ .

**3D reflection at a plane**  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

Reflections are important symmetries in physics: T (time reflection), P (reflection at a mirror), C (change of charge) are reflections. It seems today that the composition of TCP is a fundamental symmetry in nature.