| MATRICES AND GAUSS-JORDAN | Math 21b, O. Knill |
|---|---|

HOMEWORK: 1.2: 4,10,18,30,42,32*,38*,20*

MATRIX FORMULATION. Consider the system of linear equations like

$$\left| \begin{array}{rl} 3x - y - z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

It can be written as $A\vec{x} = \vec{b}$, where $A$ is a **matrix** called **coefficient matrix** and **column vectors** $\vec{x}$ and $\vec{b}$.

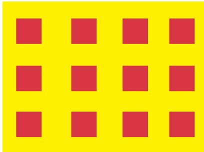
$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}.$$

($(A\vec{x})_i$ is the dot product of the $i$'th row with $\vec{x}$).

We also look at the **augmented matrix** where the last column is separated for clarity reasons.

$$B = \left[ \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

MATRIX "JARGON". A rectangular array of numbers is called a **matrix**. If the matrix has $m$ **rows** and $n$ **columns**, it is called a $m \times n$ matrix. A matrix with one column only is called a **column vector**, a matrix with one row a **row vector**. The entries of a matrix are denoted by $a_{ij}$, where $i$ is the row and $j$ is the column. In the case of the linear equation above, the matrix $A$ was a $3 \times 3$ square matrix and the augmented matrix $B$ above is a $3 \times 4$ matrix.


m

n

GAUSS-JORDAN ELIMINATION. Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling brings the matrix into **reduced row echelon form**. The elimination process consists of three possible steps which are called **elementary row operations**:

| **S**wap two rows. | **S**cale a row | **S**ubtract a multiple of a row from an other. |
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The process transfers a given matrix $A$ into a new matrix rref$(A)$

REDUCED ECHELON FORM. A matrix is called in **reduced row echelon form**

1) if a row has nonzero entries, then the first nonzero entry is 1.
2) if a column contains a leading 1, then the other column entries are 0.
3) if a row has a leading 1, then every row above has a leading 1 to the left.

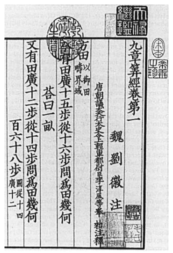
Pro memoriam: Leaders like to be first, alone of their kind and other leaders above them to their left.

RANK. The number of leading 1 in rref$(A)$ is called the rank of $A$.

SOLUTIONS OF LINEAR EQUATIONS. If $Ax = b$ is a linear system of equations with $m$ equations and $n$ unknowns, then $A$ is a $m \times n$ matrix. We see that there are the following three possibilities:

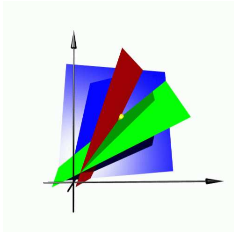
- **Exactly one solution.** There is a leading 1 in each row but not in the last row.
- **Zero solutions.** There is a leading 1 in the last row.
- **Infinitely many solutions.** There are rows without leading 1 and no leading 1 in last row.

JIUZHANG SUANSHU. The technique of successively eliminating variables from systems of linear equations is called **Gauss elimination** or **Gauss Jordan elimination** and appeared already in the Chinese manuscript "Jiuzhang Suanshu" ('Nine Chapters on the Mathematical art'). The manuscript appeared around 200 BC in the Han dynasty and was probably used as a textbook. For more history of Chinese Mathematics, see http://aleph0.clarku.edu/ ~ djoyce/mathhist/china.html.


九章筭經卷第一
魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 勅注釋
以御田疇界域
田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
百六十八步

EXAMPLES. The reduced echelon form of the augmented matrix $B$ determines on how many solutions the linear system $Ax = b$ has.

THE GOOD (**1 solution**)



$$\left[ \begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & -3 & -12 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -4 \end{array} \right]$$

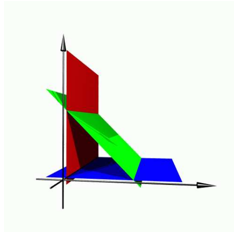
$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Rank$(A) = 3$, Rank$(B) = 3$.

THE BAD (**0 solution**)



$$\left[ \begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -7 \end{array} \right]$$

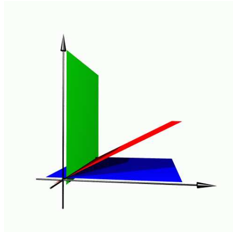
$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Rank$(A) = 2$, Rank$(B) = 3$.

THE UGLY ($\infty$ **solutions**)



$$\left[ \begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Rank$(A) = 2$, Rank$(B) = 2$.

JORDAN. The German geodesist Wilhelm **Jordan** (1842-1899) applied the Gauss-Jordan method to finding squared errors to work on surveying. (An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also (Jordan form) and is often mistakenly credited with the Gauss-Jordan process.)

GAUSS. **Gauss** developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method for solving astronomical problems. One of Gauss successes was the prediction of an asteroid orbit using linear algebra.

CERES. On 1. January of 1801, the Italian astronomer Giuseppe Piazzi (1746-1826) discovered **Ceres**, the first and until 2001 the largest known asteroid in the solar system. (A new found object called 2001 KX76 is estimated to have a 1200 km diameter, half the size of Pluto) Ceres is a rock of 914 km diameter. (The pictures Ceres in infrared light). Gauss was able to predict the orbit of Ceres from a few observations. By parameterizing the orbit with parameters and solving a linear system of equations (similar to one of the homework problems, where you will fit a cubic curve from 4 observations), he was able to derive the orbit parameters.