# CHECKLIST FOR FINAL (since 2. Midterm), Math21b, O.Knill

## Abstract

DIFFERENTIAL EQUATIONS $\dot{x}(t) = F(x(t))$ define continuous dynamical systems. For linear differential equations $\dot{x} = Ax$ the solution is the DISCRETE DYNAMICAL SYSTEM $x(t+1) = Bx(t) = e^A x(t) = Bx(t)$. LINEAR DIFFERENTIAL EQUATIONS are written as $\dot{x} = Ax$ or $p(D)f = 0$. More generally, we can look at $p(D)f = g$. To solve this, factor the polynomial $p(\lambda) = \prod_i(\lambda - \lambda_i)$ to get the homogeneous solution $f(x) = \sum_i a_i e^{\lambda_i x}$ and look then for a special solution. Also systems $\dot{x} = Ax$ are solved by DIAGONALIZATION. Each eigenvector $v_k$ satisfying $Av_k = \lambda_k v_k$ and evolves like $v_k(t) = e^{\lambda_k t} v_k$. A general initial condition $x = \sum_k a_k v_k$ evolves then like $x(t) = \sum_k a_k e^{\lambda_k t} v_k$. The same procedure solves PARTIAL DIFFERENTIAL EQUATIONS like the HEAT $u_t = D^2 u$ equation including variants like $u_t = (D^2 - D^4)u$ or WAVE EQUATION $u_{tt} = D^2 u$ including variants like $u_{tt} = (D^2 - 1)u$ where FOURIER diagonalizes $D^2$ or more generally $p(D^2)$.

## Glossary I

**COMPLEX NUMBERS** $x + iy = re^{i\theta} = r\cos(\theta) + ri\sin(\theta)$.
**LINEAR DISCRETE DYNAMICAL SYSTEM** Linear map $x \mapsto Ax$ defines system $\vec{x}(t+1) = A\vec{x}(t)$ with solution $x(t) = A^t x(0)$ for integer $t$.
**ASYMPTOTIC STABILITY** $A^n \vec{x} \to 0$ for all $\vec{x}$ rsp solution $x(t) \to 0$ for $t \to \infty$.
**LINEAR SPACE** $X$ If $x, y$ are in $X$, then $x + y, \lambda x$ are in $X$. Especially, 0 is in $X$.
**LINEAR MAP** $T(x+y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$ and $T(0) = 0$.
**DIAGONALIZATION** possible if $A$ is symmetric or if all eigenvalues are different.
**TRACE** $\mathrm{tr}(A) =$ sum of diagonal entries $= \sum_j \lambda_j$.
**DETERMINANT** $\det(A) =$ product of diagonal entries $= \prod_j \lambda_j$.
**TRACE AND DETERMINANT** Determine stability in two dimensions.
**LINEAR DIFFERENTIAL EQUATION** $\dot{x} = Ax$, where $A$ is a matrix.
**DIFFERENTIAL OPERATOR** polynomial $p(D)$ in $D$. Example $T = p(D) = D^2 + 3D, Tx = x'' + 3x'$.
**HOMOGENEOUS DIFFERENTIAL EQUATION** $p(D)f = 0$. Example: $f'' + 3f' = 0$.
**INHOMOGENEOUS DIFFERENTIAL EQUATION** $p(D)f = g$. Example: $f'' + 3f' = \sin(t)$.
**1D LINEAR DIFFERENTIAL EQUATION** $f' = \lambda f,\ f(t) = e^{\lambda t} f(0)$.
**1D DAMPED HARMONIC OSCILLATOR** $f'' + bf' + cf,\ f(t) = e^{-at}(A\cos(kt) + B\sin(kt))$ if $a \pm ik$ are roots of $\lambda^2 + b\lambda + c = 0$.
**LINEAR ODE WITH CONSTANT COEFFICIENTS** $p(D)f = g$ like $f''' - f' + 3f = \sin(t)$.
**DEGREE OF DIFFERENTIAL EQUATION** maximal number of derivatives: $f'''' + f'' = e^t$ has degree 4.
**GENERALIZED INTEGRATION** $((D - \lambda)^{-1} g)(t) = Ce^{\lambda t} + e^{\lambda x} \int_0^t e^{-\lambda s} g(s)\, ds$.
**HOMOGENEOUS LINEAR ODE** $p(D)f = 0$ like for example $f''(t) + f'(t) - 2f(t) = 0$.

## Key Points

**DIAGONALIZATION** is possible for symmetric matrices and if all eigenvalues are different.
**STABILITY** Eigenvalues determine stability.
**CLOSED SOLUTIONS** are obtained by using an eigenbasis both in discrete/continuous case.
**BASIC DIFFERENTIAL EQUATIONS** $f' = \lambda f, f'' + c^2 f = 0$.
**FOURIER BASIS** diagonalizes $D^2$. Watch even and odd case. For PdE's use sin-series only.
**NONHOMOGENEOUS SYSTEMS** are solved by operator or cookbook method.
**NONLINEAR SYSTEMS** can be understood by analyzing equilibrium points.

## Glossary II

**INNER PRODUCT** $(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$.
**LENGTH OF FUNCTION** $\sqrt{(f, f)} = ||f||$.
**FOURIER SERIES** $f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$.
**FOURIER BASIS** $1/\sqrt{2}, \cos(nx), \sin(x)$ for functions on $[-\pi, \pi]$.
**FOURIER COEFFICIENTS** $a_0 =< f, 1/\sqrt{2} >$, $a_n =< f, \cos(nx) >$, $b_n =< f, \sin(nx) >$.
**PARSEVAL** $a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$, if $f(x) = a_0/\sqrt{2} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$.
**HEAT EQUATION** $\dot{f} = \mu D^2 f$ with solution $f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$.
**WAVE EQUATION** $\ddot{f} = c^2 D^2 f$ with solution $f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc} \sin(nx)\sin(nct)$
**HEAT EQUATION VARIANT** $\dot{f} = \mu D^2 f - bf$ with solution $f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{(-n^2\mu - b)t}$.
**WAVE EQUATION VARIANT** $\ddot{f} = c^2 D^2 f - bf$ with solution $f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx) \cos(\sqrt{n^2c^2 + b}t) + \frac{b_n}{nc} \sin(nx) \sin(\sqrt{n^2c^2 + b}t)$
**STABILITY FOR DISCRETE SYSTEMS**: $|\lambda_i| < 1$.
**STABILITY FOR CONTINUOUS SYSTEMS**: $\mathrm{Re}(\lambda_i) < 0$.
**NONLINEAR DIFFERENTIAL EQUATION** $\dot{x} = f(x,y), \dot{y} = g(x,y)$.
**EQUILIBRIUM POINTS** points, where $f(x,y) = g(x,y) = 0$.
**NULLCLINES** are curves, where $f(x,y) = 0$ ($x$-nullclines) or $g(x,y) = 0$ ($y$-nullclines).
**JACOBEAN** $\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$
at equilibrium point $(x_0, y_0)$ of $\dot{x} = f(x,y), \dot{y} = g(x,y)$.

## Skills

UNDERSTAND LINEAR SPACES, LINEAR MAPS. Distinguish whether linear or not.

SOLVE DISCRETE DYNAMICAL SYSTEMS $x(n+1) = Ax(n)$. By diagonalization.

SOLVE CONTINUOUS DYNAMICAL SYSTEMS $\dot{x} = Ax$. By diagonalization.

SOLVE DIFFERENTIAL EQUATIONS $p(D)f = g$ by factoring $p$ or using "cookbook".

ASYMPTOTIC STABILITY for continuous dynamical systems (mind discrete/continuous case).

PLOT PHASE SPACE for nonlinear systems: equilibrium points, nullclines, nature of equilibrium points.

MATCH PHASE SPACE WITH SYSTEM. Both linear and nonlinear.

MAKE FOURIER SYNTHESIS of function $f(x)$ on $[-\pi, \pi]$.

KNOW FOURIER BASIS diagonalizes $D^2$ or $p(D^2)$ like $D^2 - D^4 + 1$.

APPLY PARSEVAL to relate sum of series with length of function.

SOLVE HEAT EQUATION $u_t = p(D^2)$ with given initial condition by diagonalization.

SOLVE WAVE EQUATION $u_{tt} = p(D^2)$ with given initial condition by diagonalization.