SURVIVAL KIT, FIRST HOURLY	Math 21b, O. Knill

DEFINITIONS.

- [ ] **Linear subspace** $\vec{0} \in X$, $\vec{x}, \vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in X$, $\lambda\vec{x} \in X$.
- [ ] **Matrix** A is a $n \times m$ matrix, it has m columns and n rows, maps $\mathbf{R}^m$ to $\mathbf{R}^n$.
- [ ] **Square matrix** $n \times n$ matrix, maps $\mathbf{R}^n$ to $\mathbf{R}^n$.
- [ ] **Vector** $n \times 1$ matrix = column vector, $1 \times n$ matrix = row vector.
- [ ] **Linear transformation** $T : \mathbf{R}^m \to \mathbf{R}^n$, $\vec{x} \mapsto A\vec{x}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$.
- [ ] **Column vector** of A are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
- [ ] **Linear system of equations** $A\vec{x} = \vec{b}$, n equations, m unknowns.
- [ ] **Consistent system** $A\vec{x} = \vec{b}$: for every $\vec{b}$ there is at least one solution $\vec{x}$.
- [ ] **Vector form of linear equation** $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{b}$, $\vec{v}_i$ columns of A.
- [ ] **Matrix form of linear equation** $\vec{w}_i \cdot \vec{x} = b_i$, $\vec{w}_i$ rows of A.
- [ ] **Augmented matrix of** $A\vec{x} = \vec{b}$ is the matrix $[A|b]$ which has one column more as A.
- [ ] **Coefficient matrix of** $A\vec{x} = \vec{b}$ is the matrix A.
- [ ] **Matrix multiplication** $[AB]_{ij} = \sum_k A_{ik}B_{kj}$, dot product of i-th row of A with j'th column of B.
- [ ] **Gauss-Jordan elimination** $A \to \mathrm{rref}(A)$ in row reduced echelon form.
- [ ] **Gauss-Jordan elimination steps:** SSS: Swapping rows, Scaling row, Subtracting row to other row.
- [ ] **Row reduced echelon form**: every nonzero row has leading 1, columns with leading 1 are 0 away from leading 1, every row with leading 1 has every rows above with leading 1 to the left.
- [ ] **Pivot column** column with leading 1 in $\mathrm{rref}(A)$.
- [ ] **Redundant column** column with no leading 1 in $\mathrm{rref}(A)$.
- [ ] **Rank of matrix** A. Number of leading 1 in $\mathrm{rref}(A)$. It is equal to $\dim(\mathrm{im}(A))$.
- [ ] **Nullety of matrix** A. Is defined as $\dim(\ker(A))$.
- [ ] **Kernel of linear transformation** $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
- [ ] **Image of linear transformation** $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
- [ ] **Inverse** Linear transformation satisfying $S(T(x)) = x = T(S(x))$. Corresponding matrix $B = A^{-1}$.
- [ ] **Rotation in plane** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, counterclockwise rotation by angle α.
- [ ] **Dilation in plane** $\vec{x} \mapsto \lambda\vec{x}$, also called scaling.
- [ ] **Rotation-Dilation** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, rotate by $\arctan(b/a)$.
- [ ] **Horizontal and vertical shear** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- [ ] **Shear** T leaves $\vec{v}$ invariant and $A\vec{x} - \vec{x}$ parallel to $\vec{v}$. Shear-Check: all $A\vec{x} - \vec{x}$ are parallel.
- [ ] **Reflection at line** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
- [ ] **Projection onto line containing unit vector** $T(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$.
- [ ] $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X: Every $\vec{x} \in X$ can be written as $\vec{x} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$.
- [ ] $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X: $\vec{0} = \sum_i a_i\vec{v}_i \Rightarrow a_1 = \dots = a_n = 0$.
- [ ] $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis in** X: linear independent in X and span X.
- [ ] **dimension of linear space** X number of elements in a basis of X.
- [ ] $\mathcal{B}$ **coordinates** $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors $\vec{v}_i$ as columns.
- [ ] $\mathcal{B}$ **matrix of** T The matrix is $B = S^{-1}AS$.

RESULTS.

- [ ] **Linear transformations**. T is linear: $T(\vec{0}) = \vec{0}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$
- [ ] **Solution**. A linear system of equations has either exactly 1, no or infinitely many solutions.
- [ ] **Dimension formula**. $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = m$, where A is $n \times m$ matrix.
- [ ] **Behavior of kernel under elimination** kernel stays invariant under Gauss-Jordan elimination.
- [ ] **Behavior of image under elimination** image in general changes during Gauss-Jordan elimination.
- [ ] **Basis of image of** A pivot columns of A form a basis of the image of A.
- [ ] **Basis of kernel of** A introduce free variables for each non-Pivot column of A.
- [ ] **Inverse of** 2×2 **matrix** switch diagonal elements, change sign of wings and divide by determinant.
- [ ] **Kernel of composition** kernel of A is contained in the kernel of BA.
- [ ] **Image of composition** image of BA is contained in the image of B.
- [ ] **Matrix algebra** $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, etc. Note: $AB \neq BA$ in general.
- [ ] **A invertible** $\Leftrightarrow \mathrm{rref}(A) = I_n \Leftrightarrow$ columns form basis $\Leftrightarrow \mathrm{rank}(A) = n, \Leftrightarrow \mathrm{nullity}(A) = 0$.