

DIFFERENTIAL EQUATIONS $\dot{x}(t) = F(x(t))$ define continuous dynamical systems. For linear differential equations $\dot{x} = Ax$ the solution is the DISCRETE DYNAMICAL SYSTEM $x(t+1) = Bx(t) = e^A x(t) = Bx(t)$. LINEAR DIFFERENTIAL EQUATIONS are written as $\dot{x} = Ax$ or $p(D)f = 0$. More generally, we can look at $p(D)f = g$. To solve this, factor the polynomial $p(\lambda) = \prod_i (\lambda - \lambda_i)$ to get the homogeneous solution $f(x) = \sum_i a_i e^{\lambda_i x}$ and look then for a special solution. Also systems $\dot{x} = Ax$ are solved by DIAGONALIZATION. Each eigenvector v_k satisfying $Av_k = \lambda_k v_k$ and evolves like $v_k(t) = e^{\lambda_k t} v_k$. A general initial condition $x = \sum_k a_k v_k$ evolves then like $x(t) = \sum_k a_k e^{\lambda_k t} v_k$. The same procedure solves PARTIAL DIFFERENTIAL EQUATIONS like the HEAT $\dot{f} = D^2 f$ or WAVE EQUATION $\ddot{f} = D^2 f$, where FOURIER diagonalizes D^2 .

Glossary

COMPLEX NUMBERS $x + iy = r e^{i\theta} = r \cos(\theta) + ri \sin(\theta)$.
LINEAR DISCRETE DYNAMICAL SYSTEM Linear map $x \mapsto Ax$ defines orbit $\vec{x}(t+1) = A\vec{x}(t)$.
ASYMPTOTIC STABILITY $A^n \vec{x} \rightarrow 0$ for all \vec{x} rsp solution $x(t) \rightarrow 0$ for $t \rightarrow \infty$.
LINEAR SPACE X If x, y are in X , then $x + y, \lambda x$ are in X . Especially, 0 is in X .
LINEAR MAP $T(x+y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$ and $T(0) = 0$.
DIAGONALIZATION possible if A is symmetric or if all eigenvalues are different.
TRACE. $\text{tr}(A) = \text{sum of diagonal entries}, \sum_j \lambda_j$.
DETERMINANT. $\det(A) = \text{product of diagonal entries}, \prod_j \lambda_j$.
TRACE AND DETERMINANT. Determine stability in two dimensions.
LINEAR DIFFERENTIAL EQUATION $\dot{x} = Ax$, where A is a matrix.
DIFFERENTIAL OPERATOR polynomial in D . Example $T = p(D) = D^2 + 3D, Tx = x'' + 3x'$.
HOMOGENEOUS DIFFERENTIAL EQUATION $p(D)f = 0$. Example: $f'' + 3f' = 0$.
INHOMOGENEOUS DIFFERENTIAL EQUATION $p(D)f = g$. Example: $f'' + 3f' = \sin(t)$.
1D LINEAR DIFFERENTIAL EQUATION $f' = \lambda f, f(t) = e^{\lambda t} f(0)$.
1D DAMPED HARMONIC OSCILLATOR $f'' + bf' + cf, f(t) = e^{-bt}(A \cos(kt) + B \sin(kt))$ if $b \pm ik$ are eigenvalues.
LINEAR ODE WITH CONSTANT COEFFICIENTS $p(D)f = g$.
GENERALIZED INTEGRATION $((D - \lambda)^{-1} f)(x) = e^{\lambda x} (\int_0^x f(t) e^{-\lambda t} dt + C)$.
HOMOGENEOUS LINEAR ODE $p(D)f = 0$.
INNER PRODUCT $(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$.
LENGTH $\sqrt{(f, f)} = \|f\|$.
FOURIER SERIES $f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$.
FOURIER BASIS $1/\sqrt{2}, \cos(nx), \sin(x)$ for 2π -periodic functions.
FOURIER COEFFICIENTS $a_0 = (f, 1/\sqrt{2}), a_n = (f, \cos(nx)), b_n = (f, \sin(nx))$.
PERSEVAL $a_0^2/2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = |f|^2$, if $f(x) = a_0/\sqrt{2} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$.
HEAT EQUATION $\dot{f} = \mu D^2 f$ with solution $f(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$.
WAVE EQUATION $\ddot{f} = c^2 D^2 f$ with solution $f(x, t) = \sum_{n=1}^{\infty} a_n \sin(nx) \cos(nct) + \frac{b_n}{nc} \sin(nx) \sin(nct)$
STABILITY FOR DISCRETE 2×2 SYSTEMS: $|\lambda_i| < 1$.
STABILITY FOR CONTINUOUS 2×2 SYSTEMS: $\text{Re}(\lambda_i) < 0$.
NONLINEAR DIFFERENTIAL EQUATION $\dot{x} = f(x, y), \dot{y} = g(x, y)$.
EQUILIBRIUM POINTS points, where $f(x, y) = g(x, y) = 0$.
NULLCLINES are curves, where $f(x, y) = 0$ (x -nullclines) or $g(x, y) = 0$ (y -nullclines).
JACOBEAN $\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$ at critical point (x_0, y_0) of $(\dot{x} = f(x, y), \dot{y} = g(x, y))$.

Skills

UNDERSTAND LINEAR SPACES, LINEAR MAPS.
 SOLVE DISCRETE DYNAMICAL SYSTEMS $x(n+1) = Ax(n)$. By diagonalization.
 SOLVE CONTINUOUS DYNAMICAL SYSTEMS $\dot{x} = Ax$. By diagonalization.
 SOLVE DIFFERENTIAL EQUATIONS $p(D)f = g$ by factoring p or using cookbook approach
 ASYMPTOTIC STABILITY for discrete dynamical systems.
 ASYMPTOTIC STABILITY for continuous dynamical systems.
 PLOT PHASE SPACE for nonlinear systems: equilibrium points, nullclines, nature of equilibrium points.
 MATCH PHASE SPACE WITH SYSTEM.
 MAKE FOURIER SYNTHESIS of function $f(x)$ on $[-\pi, \pi]$.
 DIAGONALIZATION of D^2 by Fourier basis.
 SOLVE HEAT EQUATION with given initial condition by diagonalization.
 SOLVE WAVE EQUATION with given initial condition by diagonalization.