Math21b

Review to second midterm

Spring 2007

Oliver Knill

Orthonormal basis

$\{ w_1, w_2, \ldots, w_n \}$

orthonormal basis

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} / 2$$

orthogonal matrix

$$A^T A = I$$

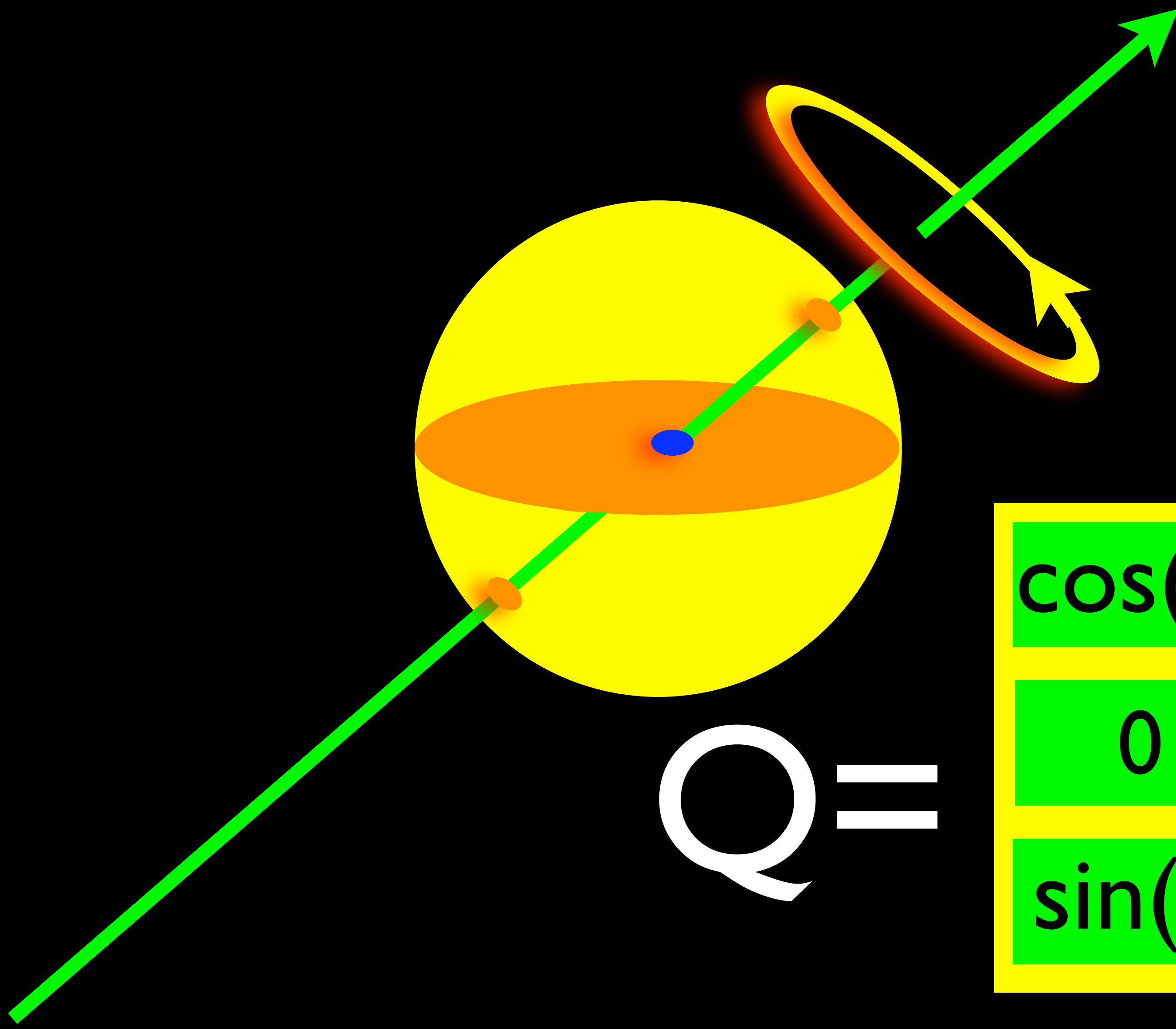
Orthogonal transformation

$$Q^T Q = I$$

- typically, reflections or rotations
- preserve length
- preserve angles
- column vectors of Q form orthonormal basis

Rotations

$\det(Q) = 1$



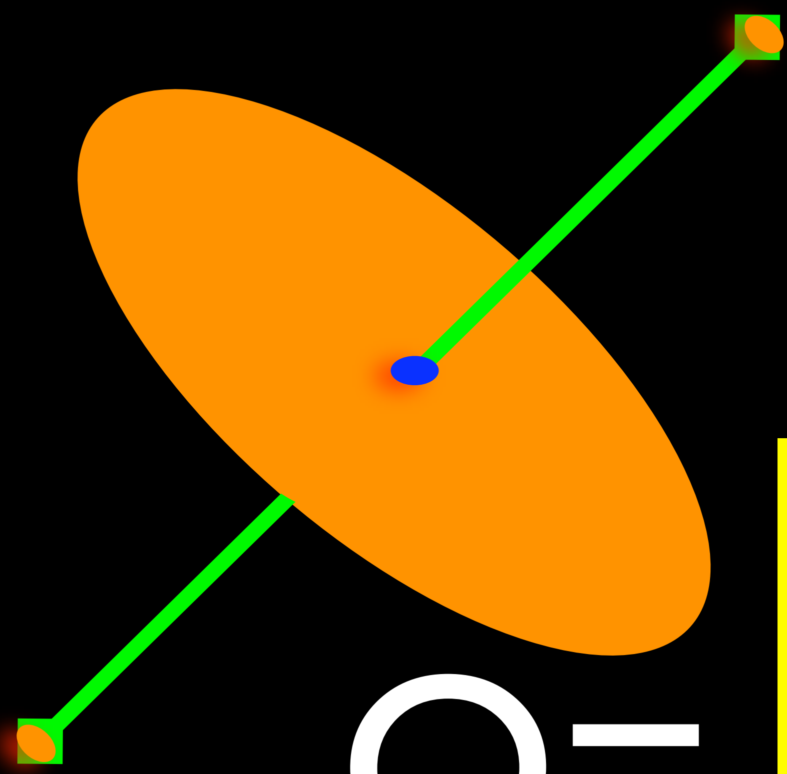
example:

$$Q = \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & 1 & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}$$

Reflections

$\det(Q)=-1$

if reflected at odd dimensional space



$\det(Q)=+1$

if reflected at even dimensional space

example:

$$Q=\begin{pmatrix}\cos(t) & 0 & \sin(t)\\ 0 & 1 & 0\\ \sin(t) & 0 & -\cos(t)\end{pmatrix}$$

don't confuse orthogonal transformations
with orthogonal projections
which satisfy for n x n matrices

$$A^2 = A$$

and are in general not
invertible.

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is an orthogonal matrix?

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} /2$$

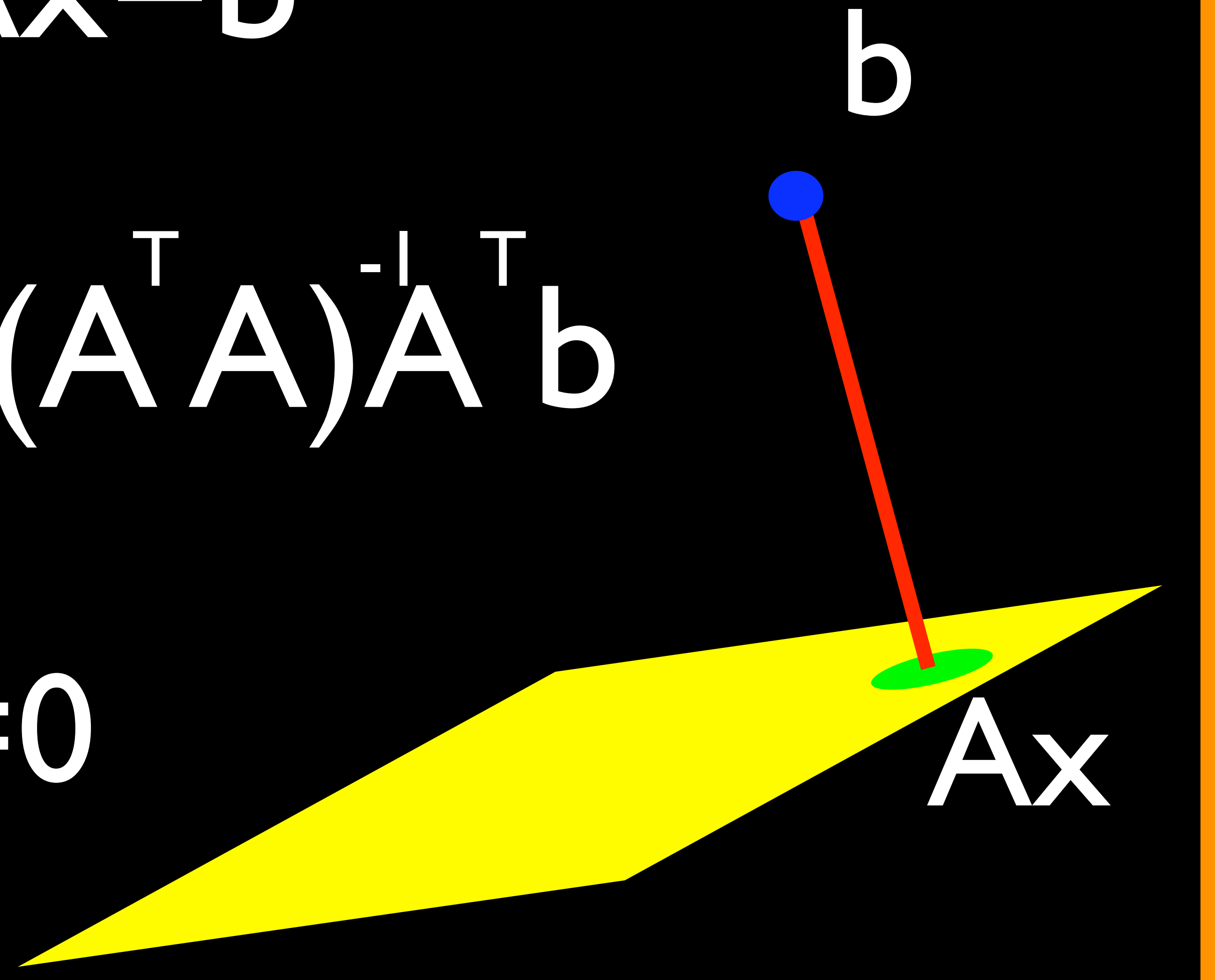
$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} /2$$

Least Square Solution

$$Ax=b$$

$$x_* = (A^T A)^{-1} A^T b$$

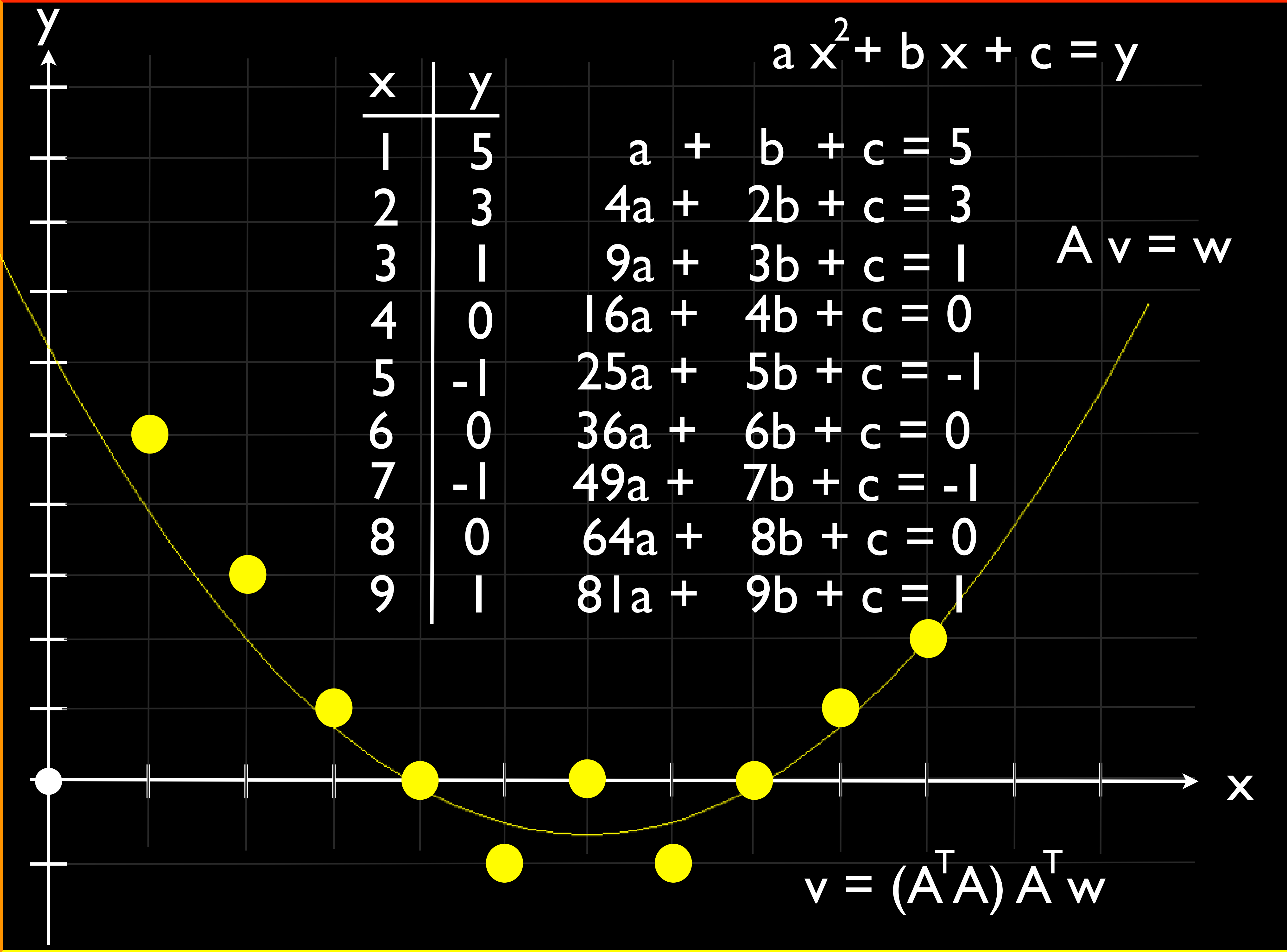
$$A^T(Ax-b)=0$$



Datafitting

To fit data with a set of functions, just plug in the data sp that all data satisfy the equations.

Here is an example with functions $1, x, x^2$



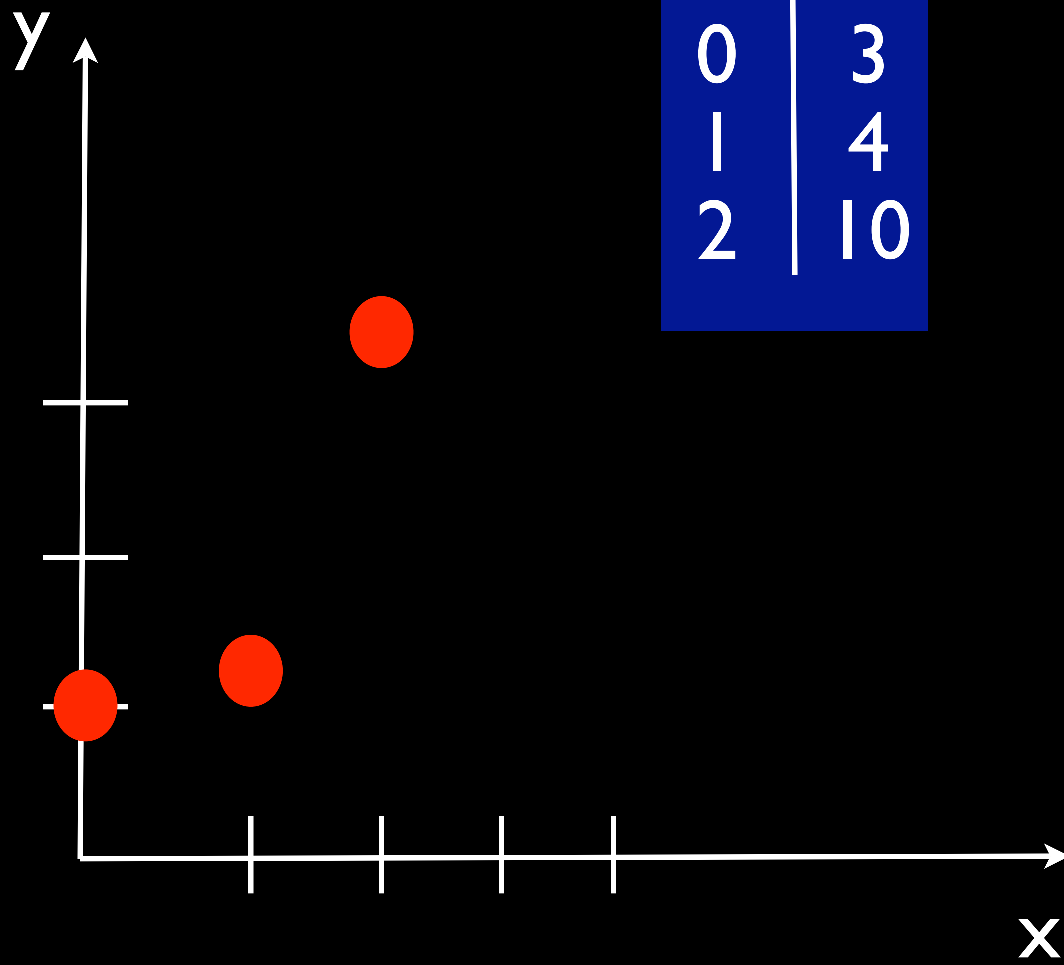
y
x
$a x^2 + b x + c = y$
x | y
1 | 5
2 | 3
3 | 1
4 | 0
5 | -1
6 | 0
7 | -1
8 | 0
9 | 1
$a + b + c = 5$
$4a + 2b + c = 3$
$9a + 3b + c = 1$
$16a + 4b + c = 0$
$25a + 5b + c = -1$
$36a + 6b + c = 0$
$49a + 7b + c = -1$
$64a + 8b + c = 0$
$81a + 9b + c = 1$
$A v = w$
$v = (A^T A) A^T w$

blackboard problem

Fit the following data with functions

$\cos(\pi x/2)$

$x+1$



x	y
0	3
1	4
2	10

Solution:

x	y
0	3
1	4
2	10

$$a\cos(\pi x/2) + b(x+1) = y$$

$$a\,1 + b\,1 = 3$$
$$a\,0 + b\,2 = 4$$
$$a\,(-1) + b\,3 = 10$$

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}$$

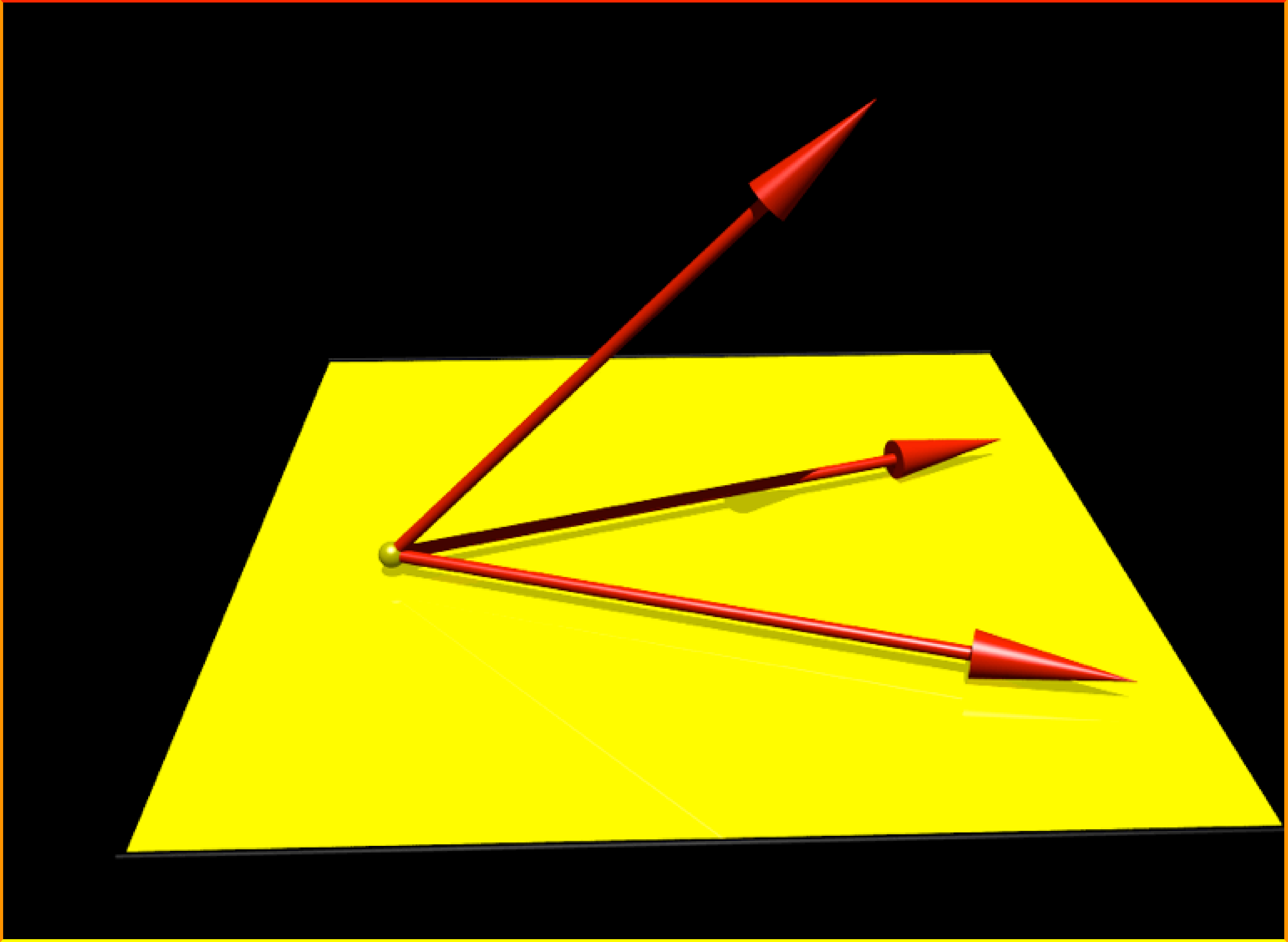
$$A^T b = \begin{bmatrix} -7 \\ 41 \end{bmatrix}$$

$$(A^T A)^{-1} A^T b = \begin{bmatrix} -2/3 \\ 17/6 \end{bmatrix}$$

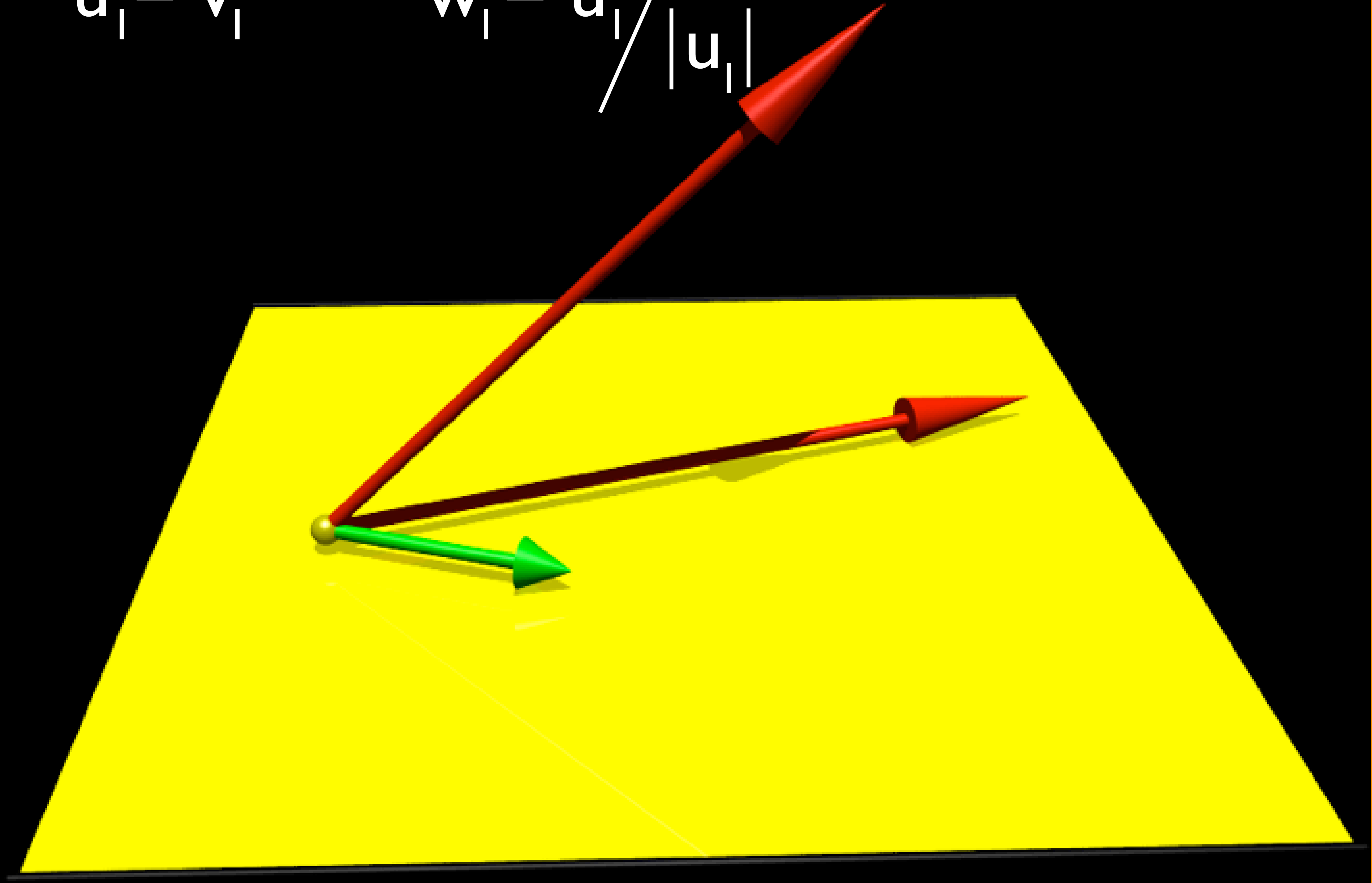
$$A^T A = \begin{bmatrix} 2 & -2 \\ -2 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix} /12$$

Gram Schmidt
+QR Decomposition

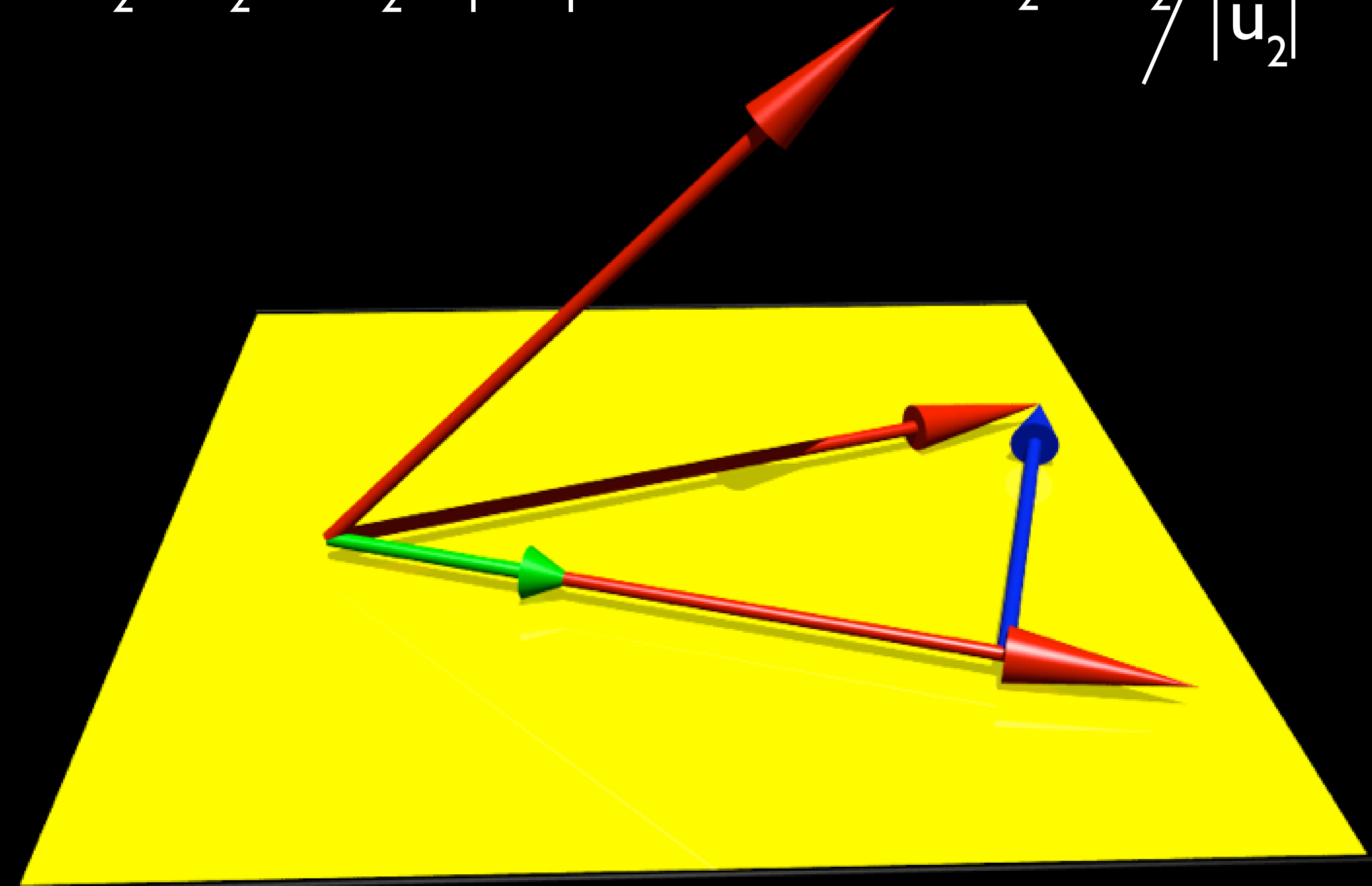


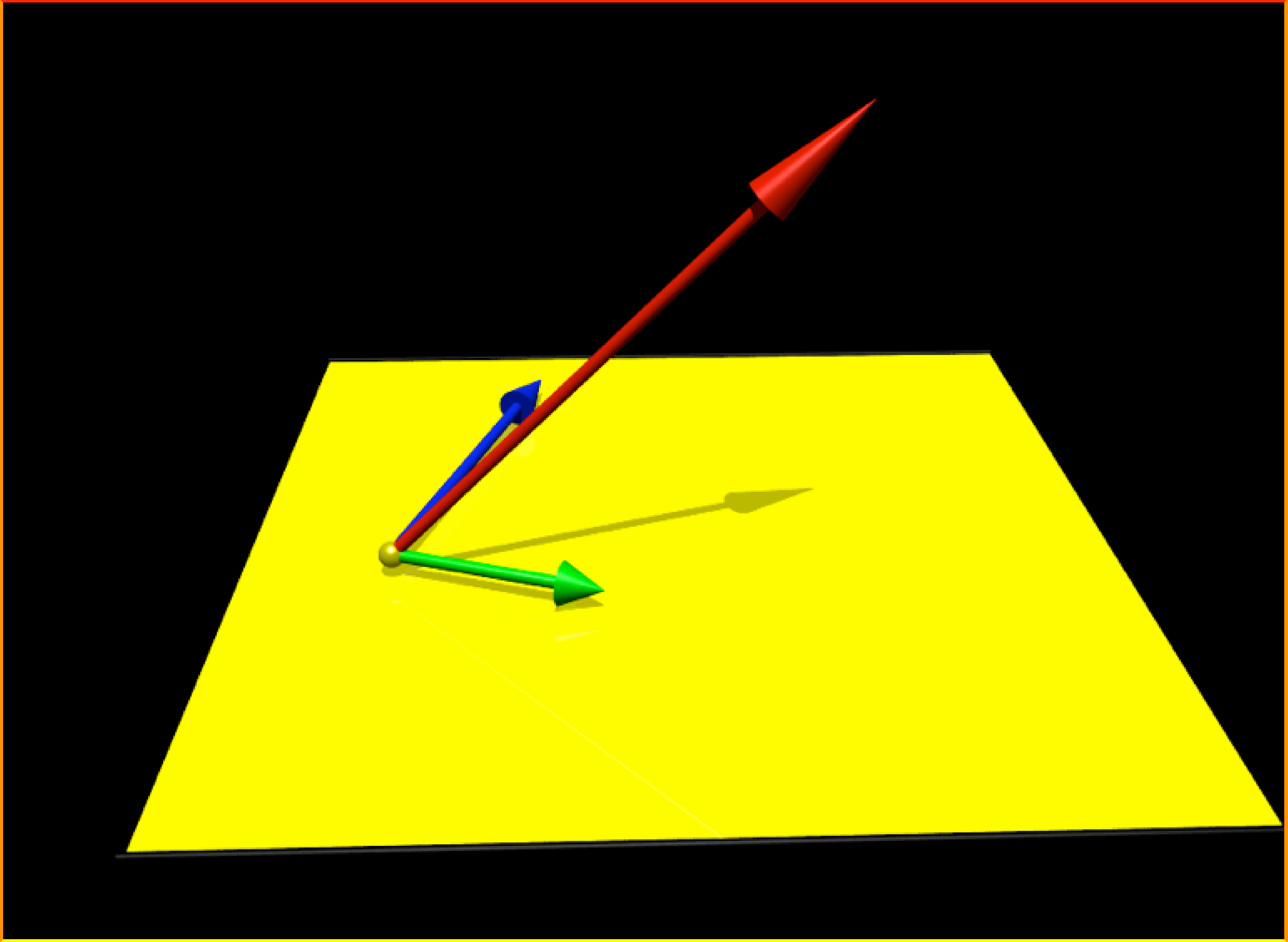
$u_1 = v_1$ $w_1 = u_1 / |u_1|$



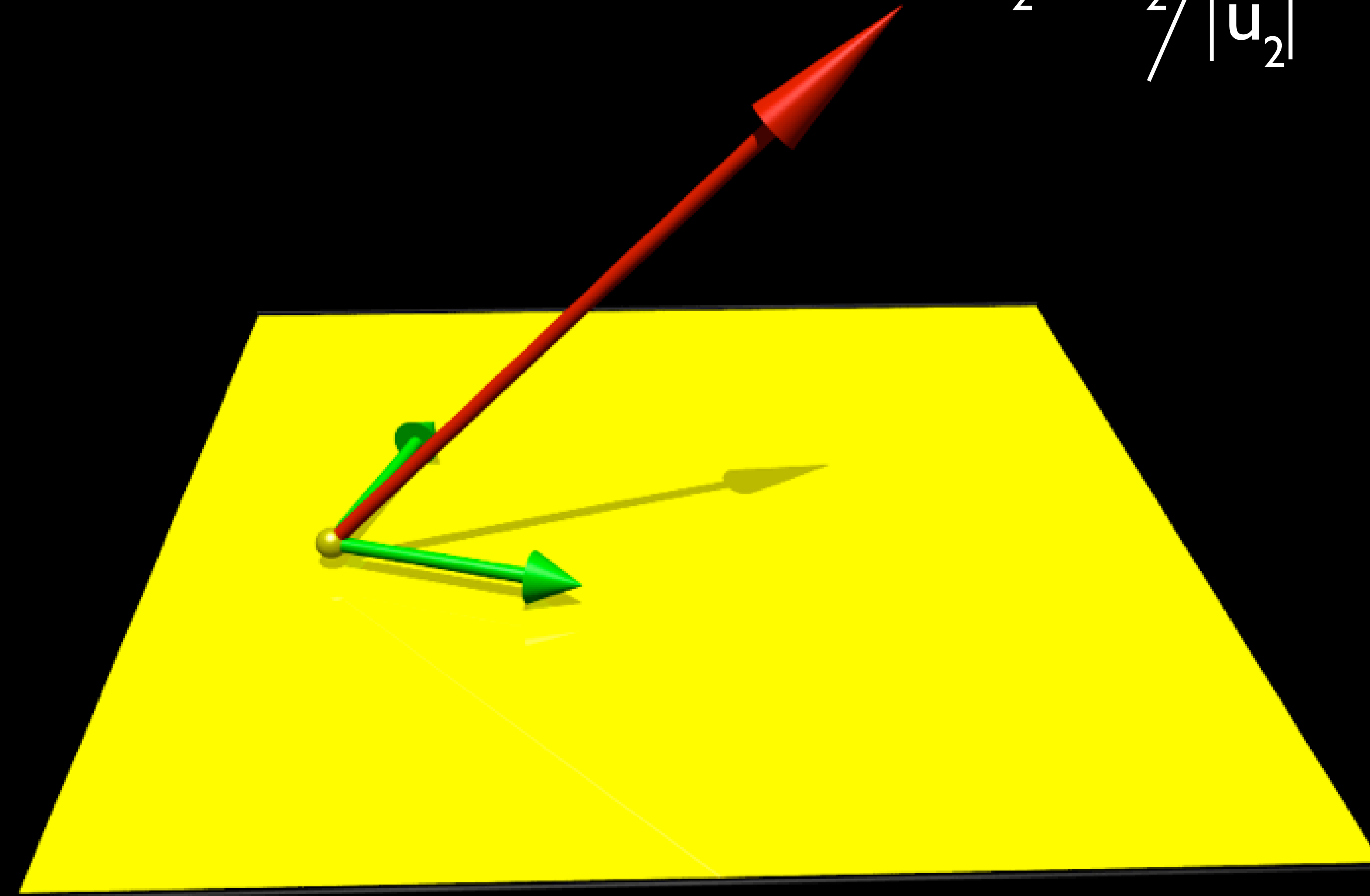
$$u_2 = v_2 - (v_2 . w_1)\, w_1$$

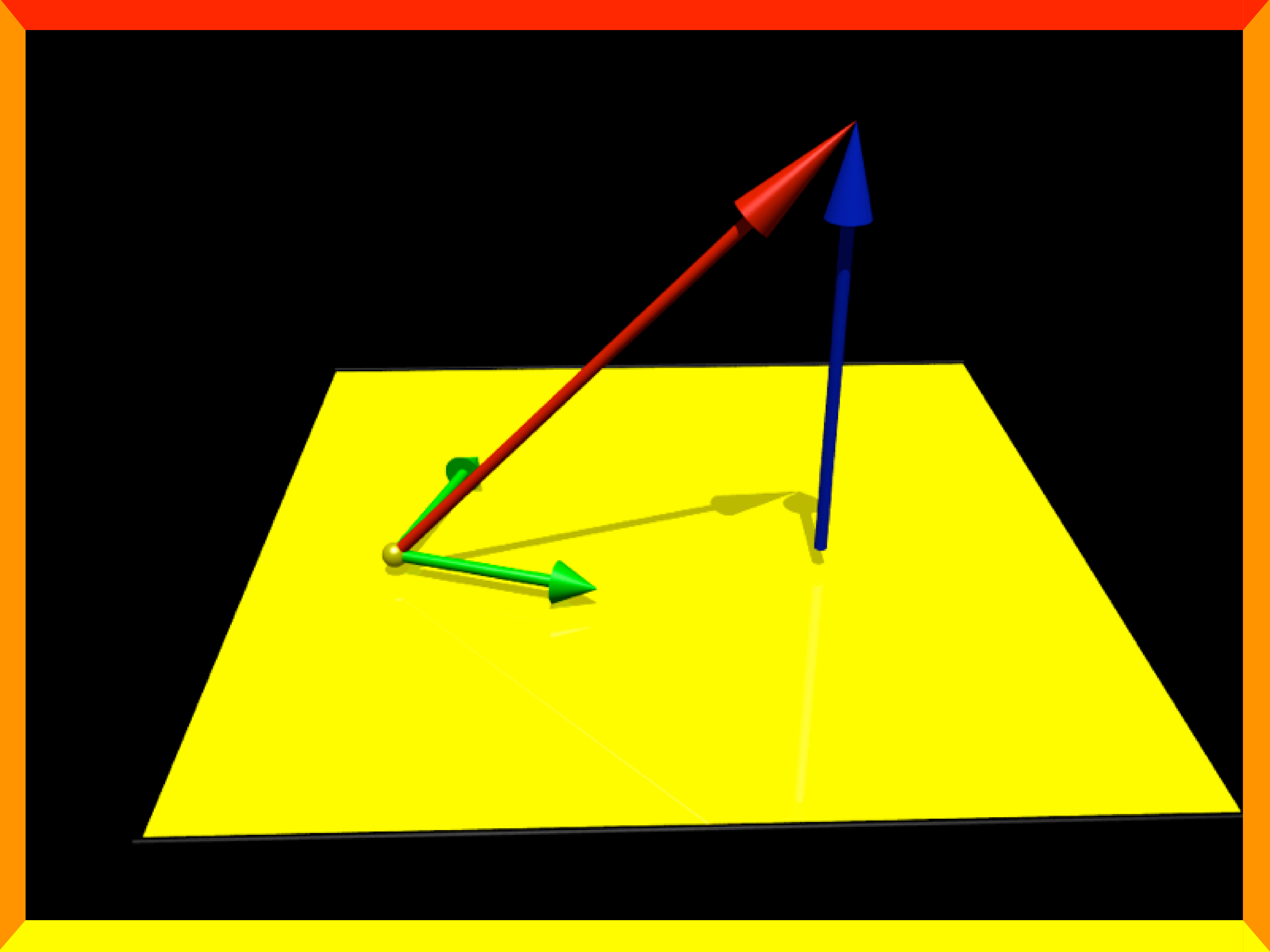
$$w_2 = u_2 / |u_2|$$



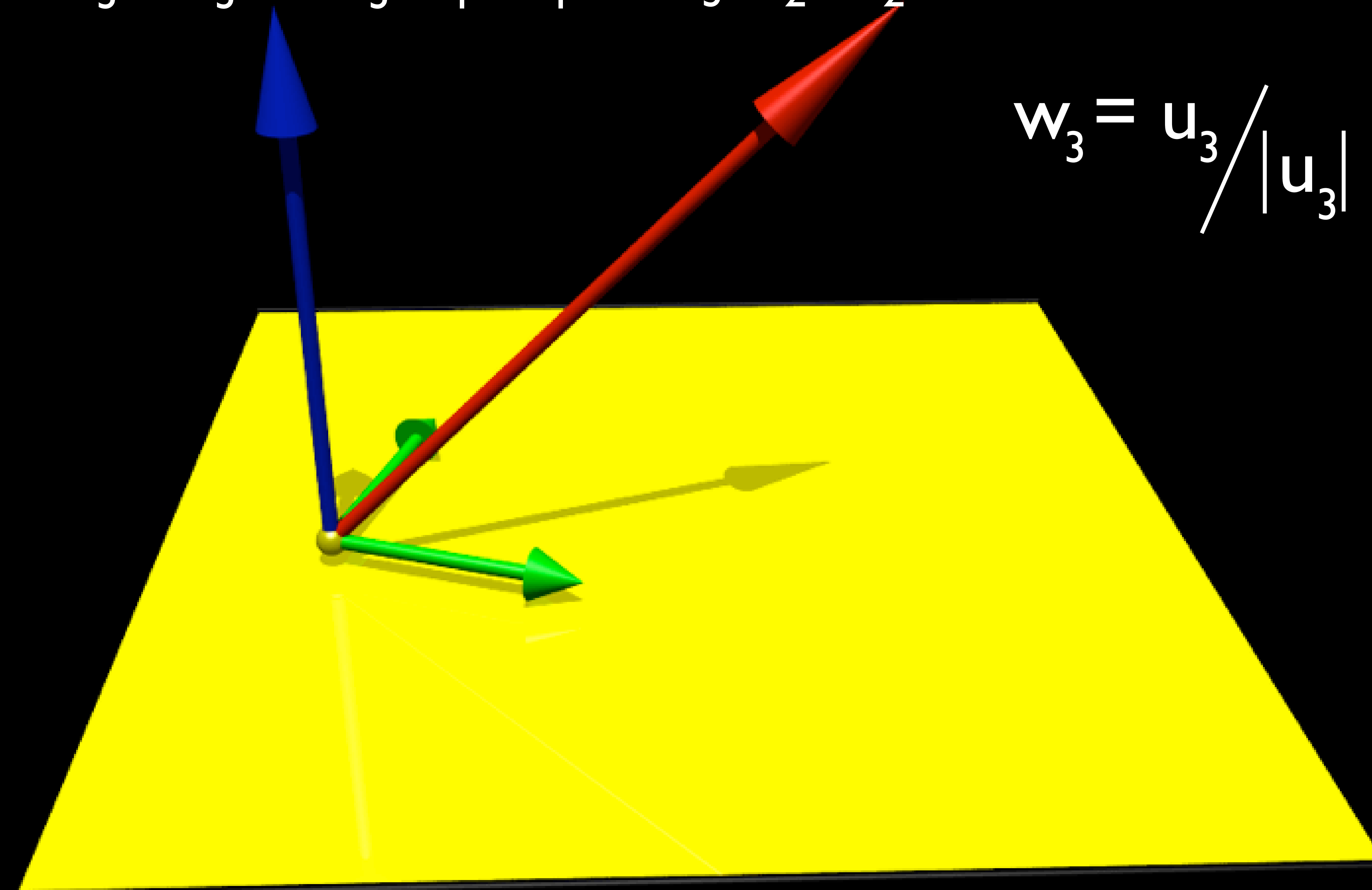


$$w_2 = u_2 / |u_2|$$

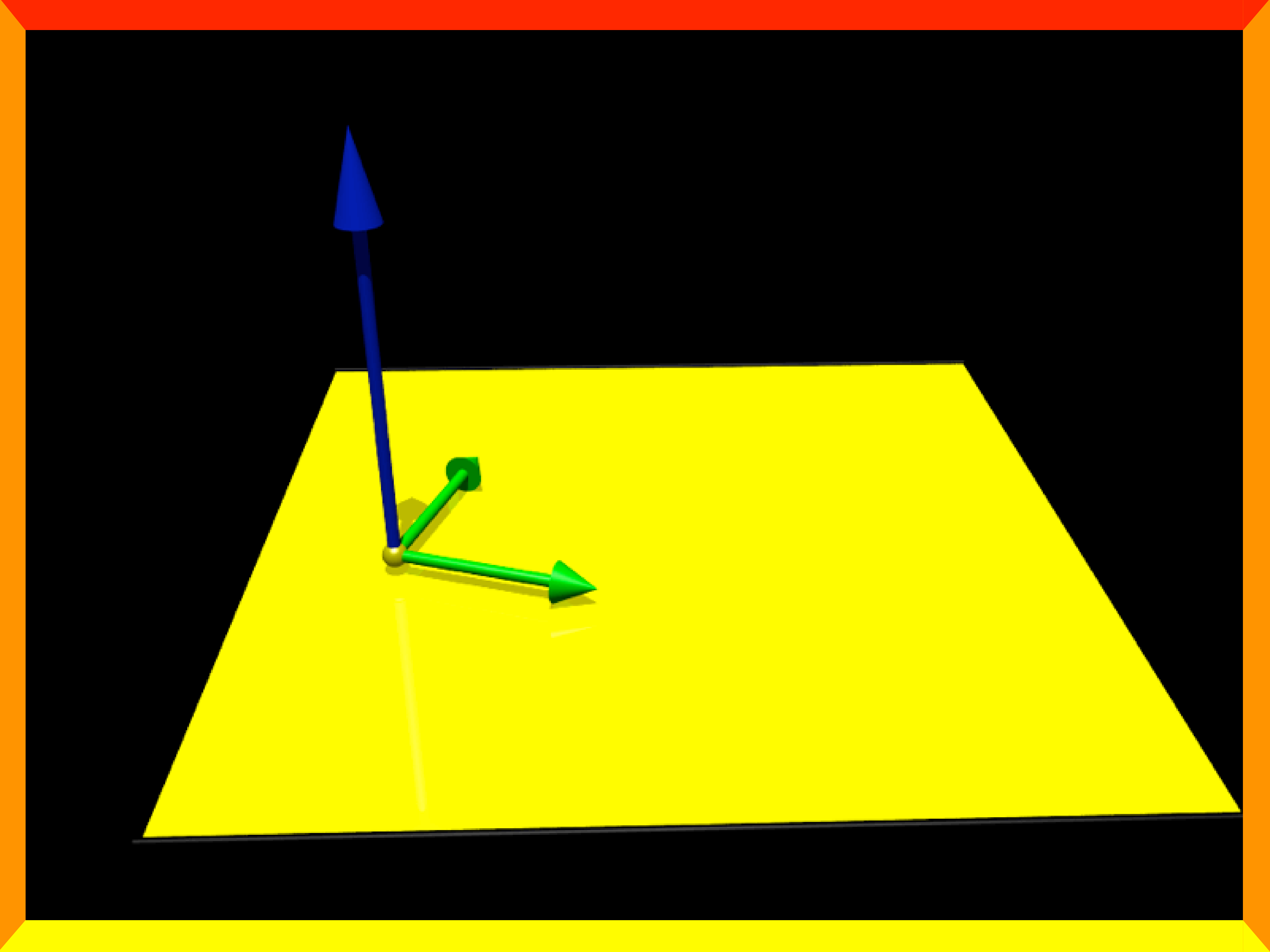


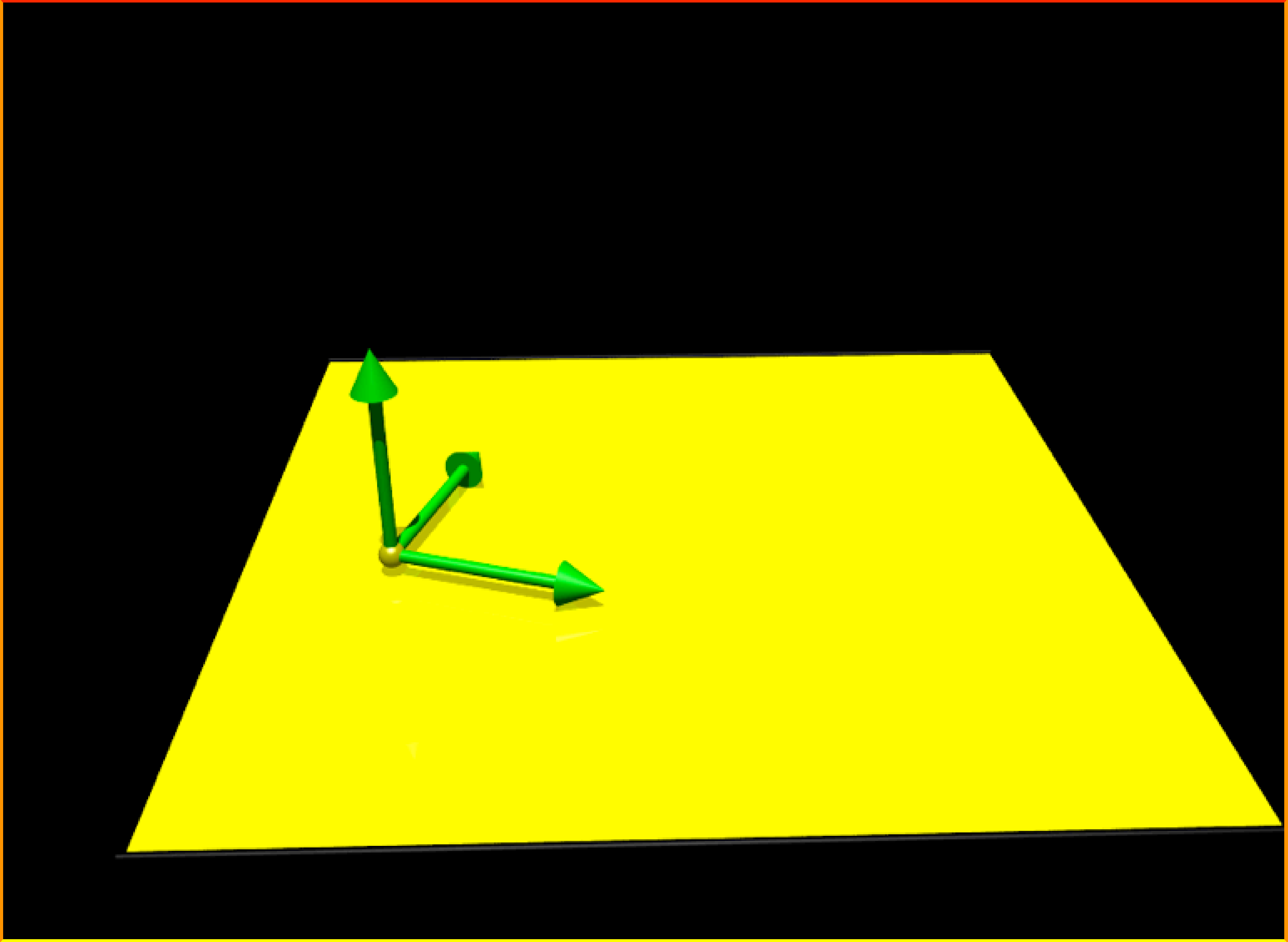


$$u_3 = v_3 - (v_3 . w_1)\, w_1 - (v_3 . w_2)\, w_2$$



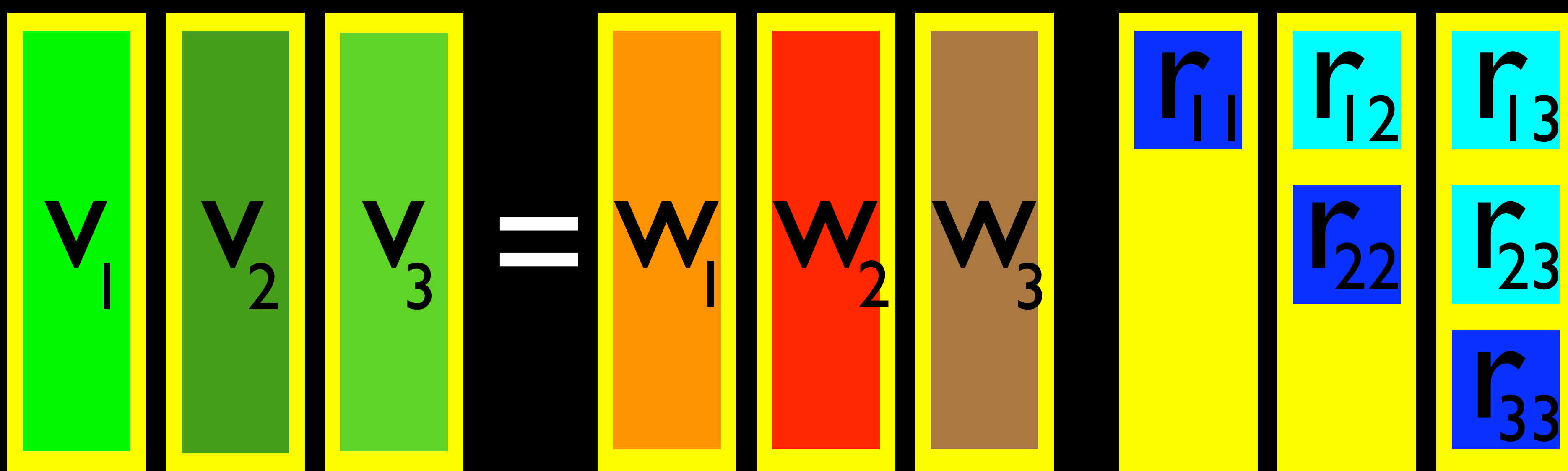
$$w_3 = u_3 / |u_3|$$





$$A = Q R$$

$$r_{ii} = |u_i| \qquad r_{ij} = (w_i \cdot v_j)$$



blackboard problem

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Apply Gram-Schmidt to this basis and do QR decomposition.

Determinants

$$\det\begin{pmatrix} a \end{pmatrix} = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

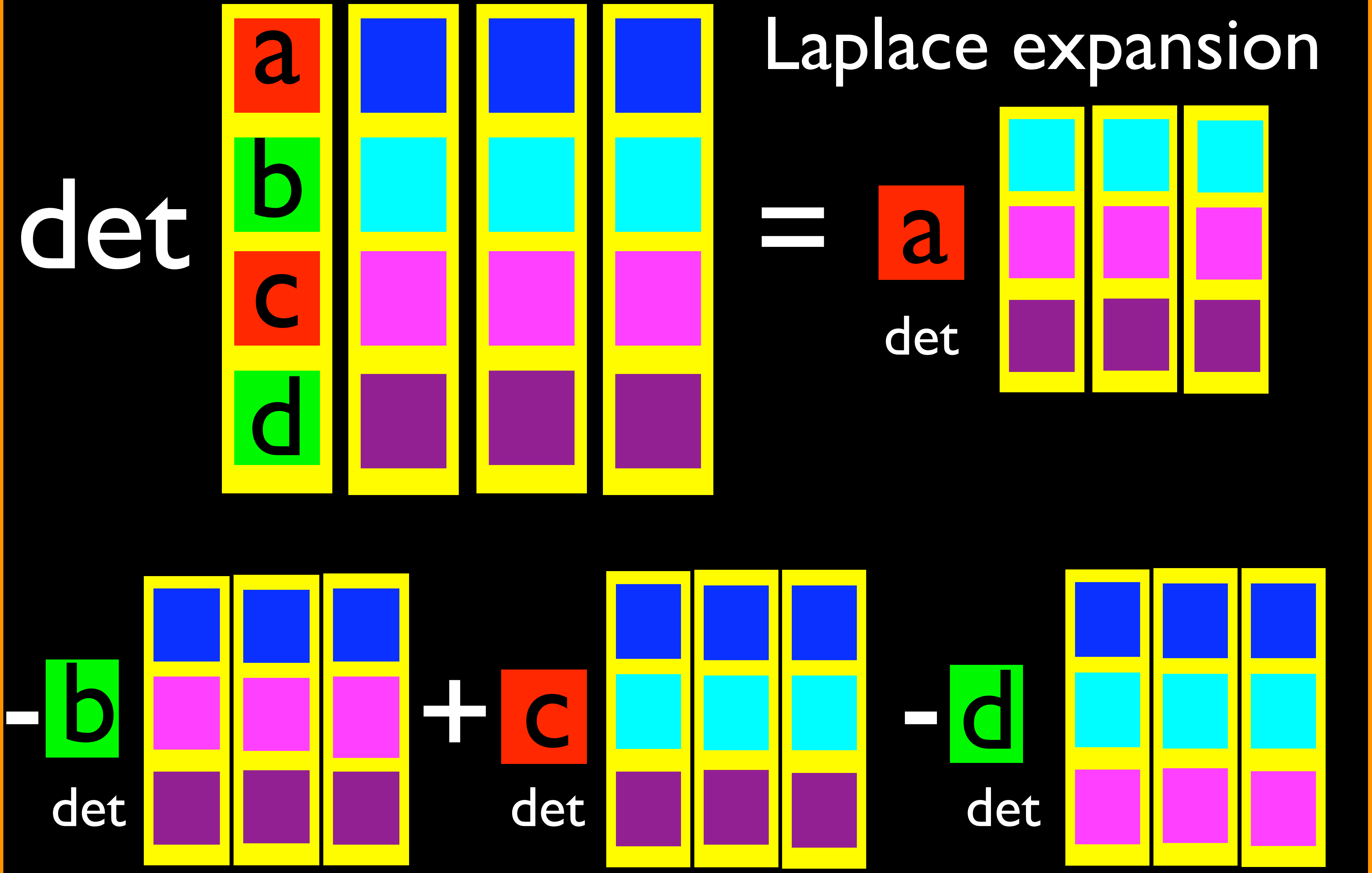
$$+ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad - \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$aei+bfg+cdh \;-\; bdi+ceg+ahf$$

which by the way can be remembered much better with the following poem:

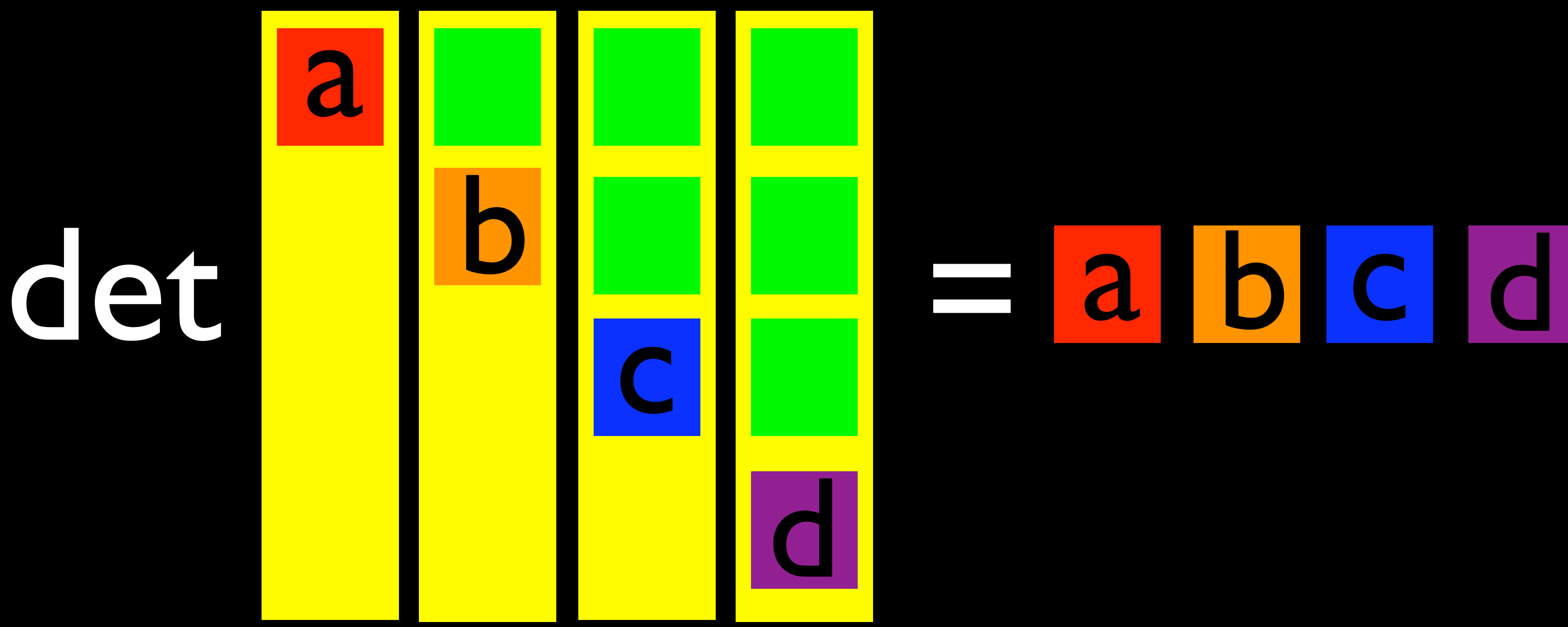
$$\det\begin{pmatrix} a & e & u \\ n & g & l \\ i & o & k \end{pmatrix} =$$

gak+onu+lie - alo-ken-gui

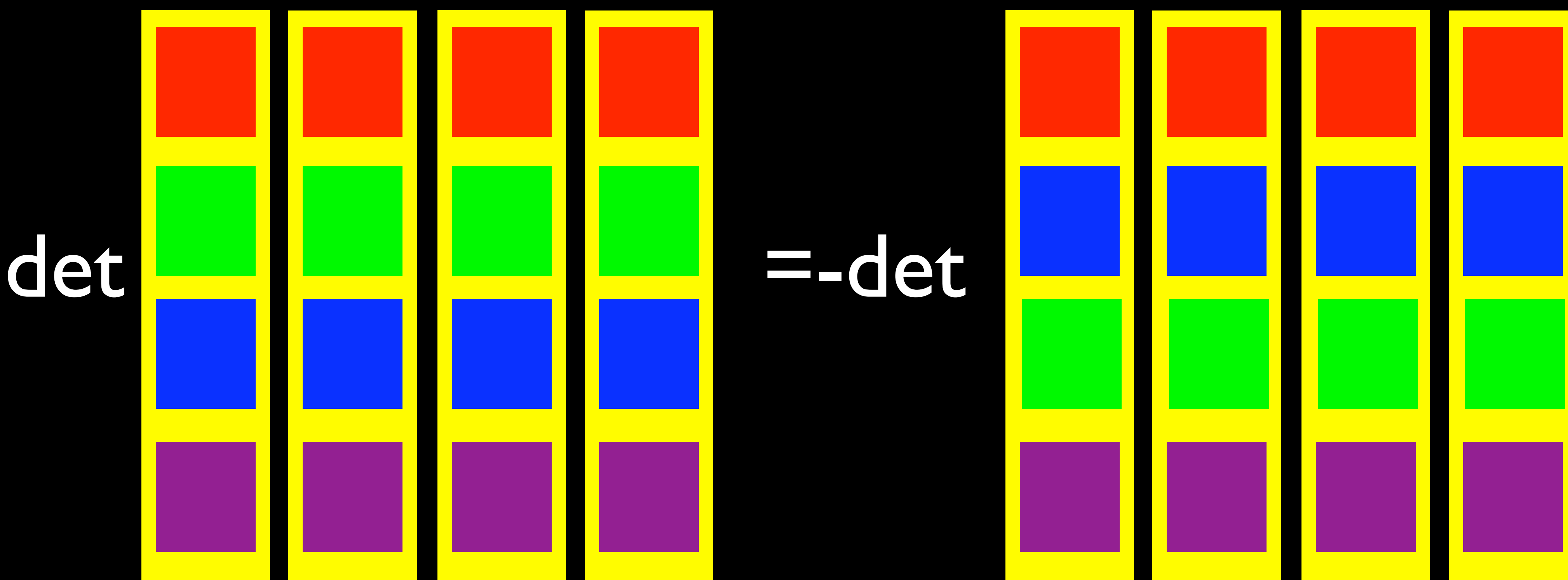

Laplace expansion
det
a
b
c
d
=
a
det
-b
det
+c
det
-d
det

$$\det\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B)$$

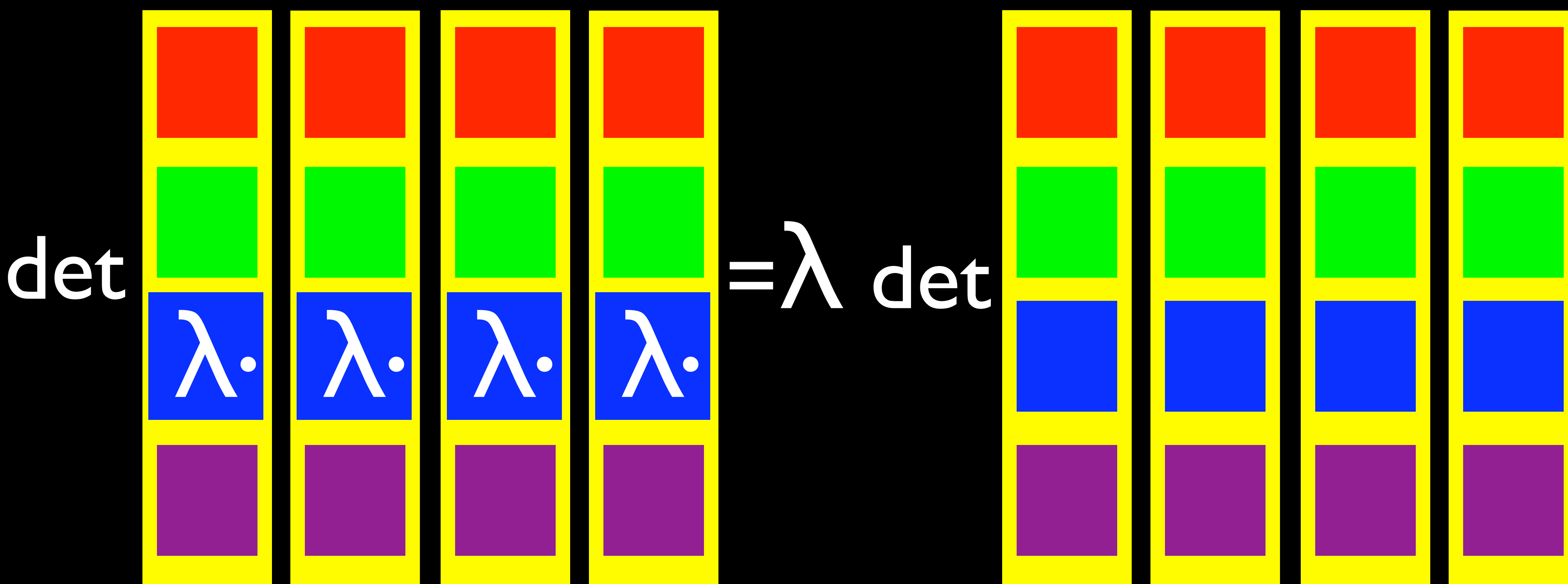
upper or lower triangular matrices



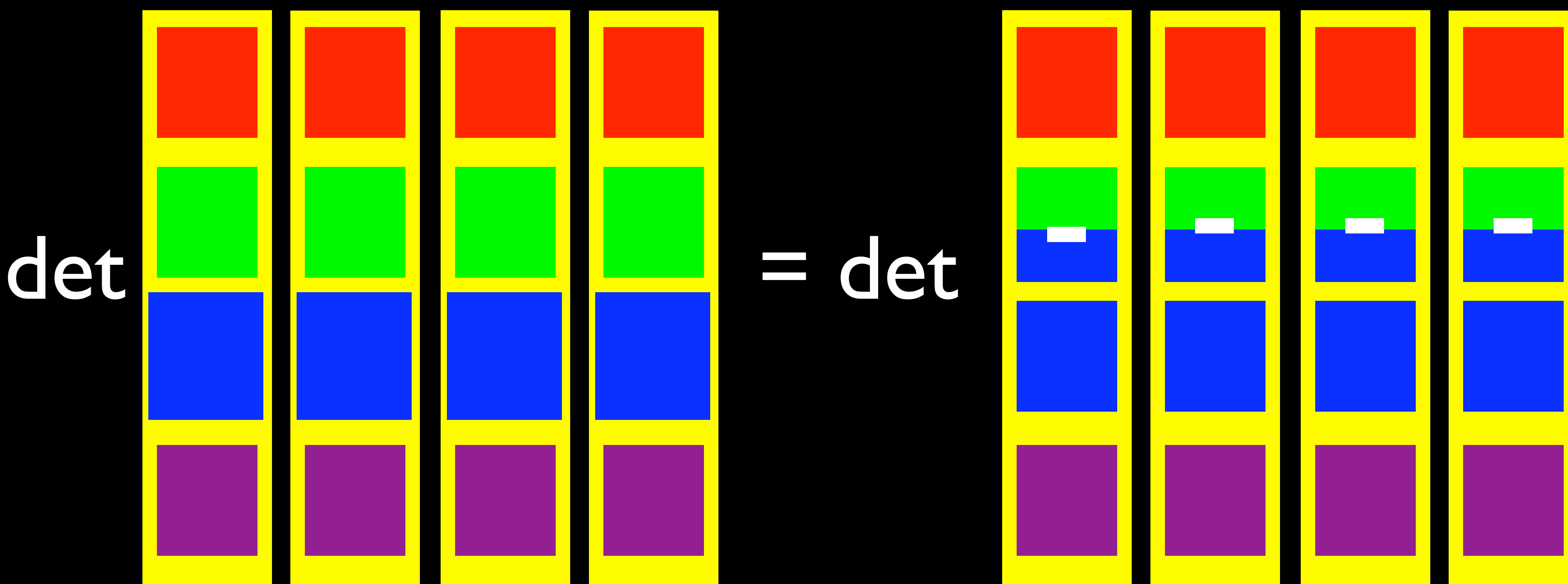
row reduction: swap



row reduction: scale



row reduction: subtract



blackboard problem

What is the determinant of

2	0	3	4	5	2
1	2	3	3	4	3
4	0	2	2	3	4
0	0	0	3	1	1
0	0	0	2	2	1
0	0	0	1	2	1


DVD
to win

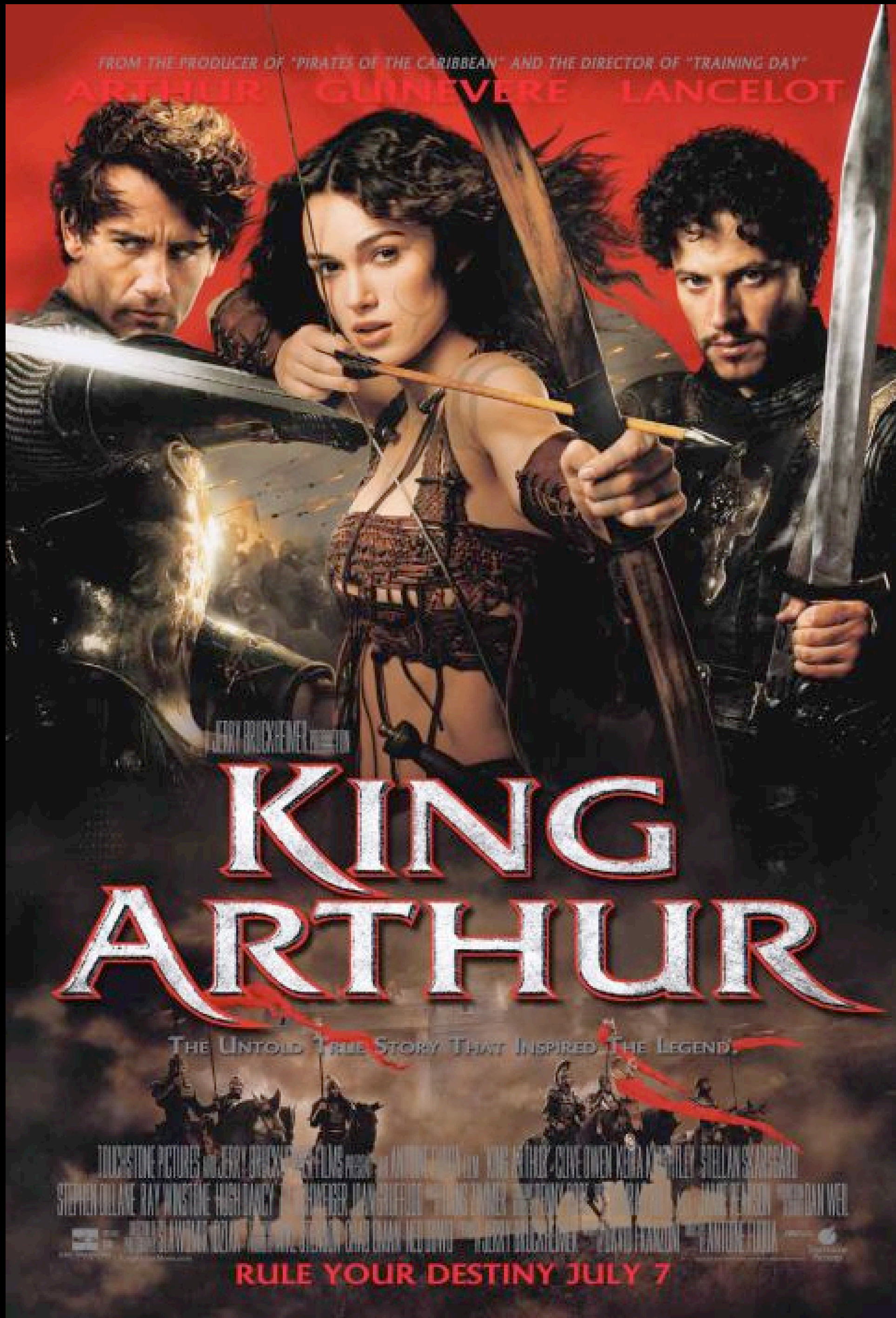
Its not the famous Monty Python movie:

but a decent Hollywood adaption of the story

comparison

original
Monty
Python
battle

adaptation
of
Hollywood

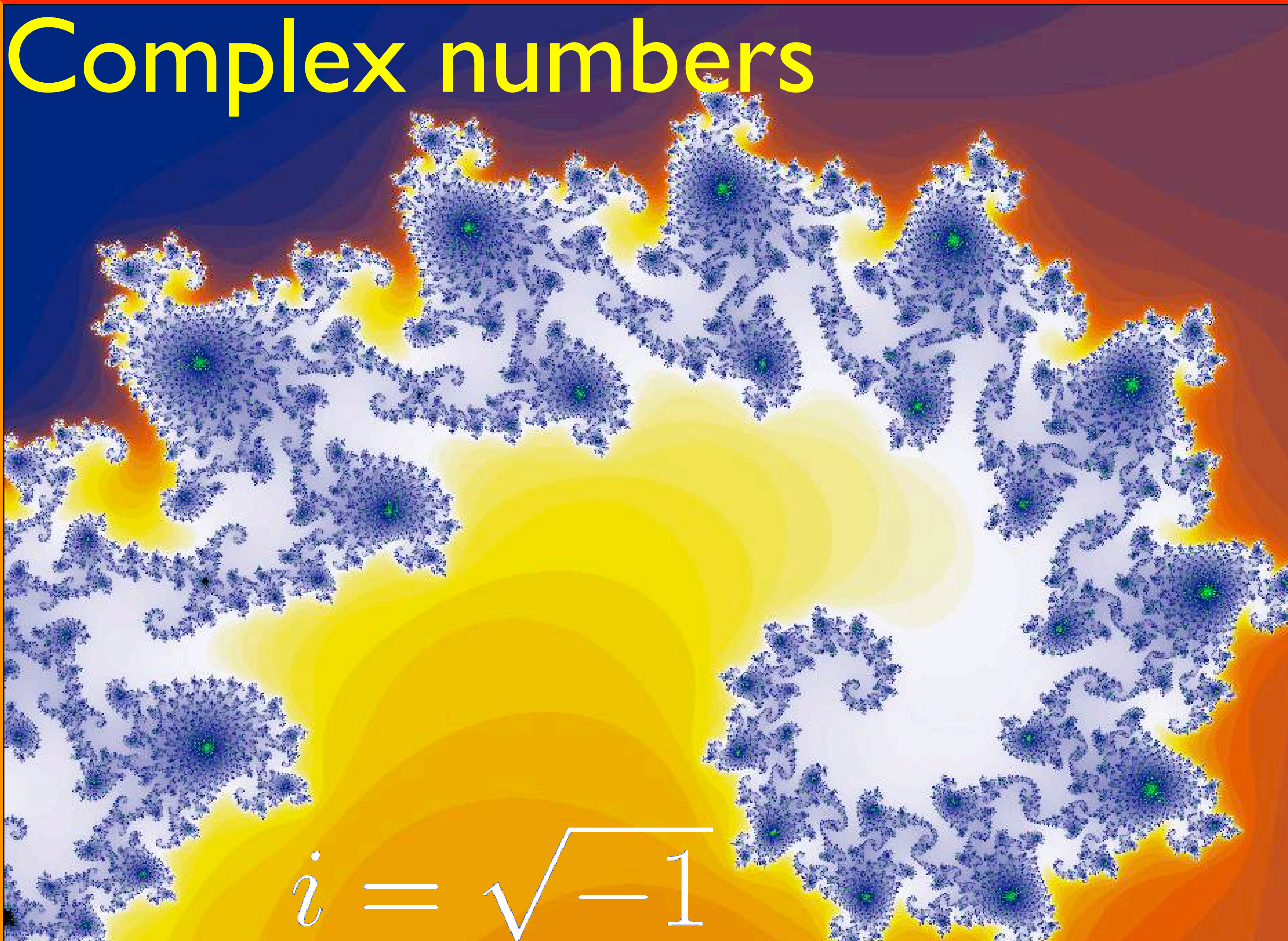


ready?
the next slide
gives the
problem

What is the determinant of

2	2	0	0	0	0
3	2	2	0	0	0
4	3	3	2	0	0
4	3	3	3	2	0
3	2	2	2	2	2
2	0	0	0	0	0

Complex numbers



$$i = \sqrt{-1}$$

Complex numbers

$$i = \sqrt{-1}$$

Gauss in 1825 : “The true metaphysics of the square root of -1 is elusive”.

Euler Formula

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

Is the gateway to most secrets in complex numbers.

Proof:

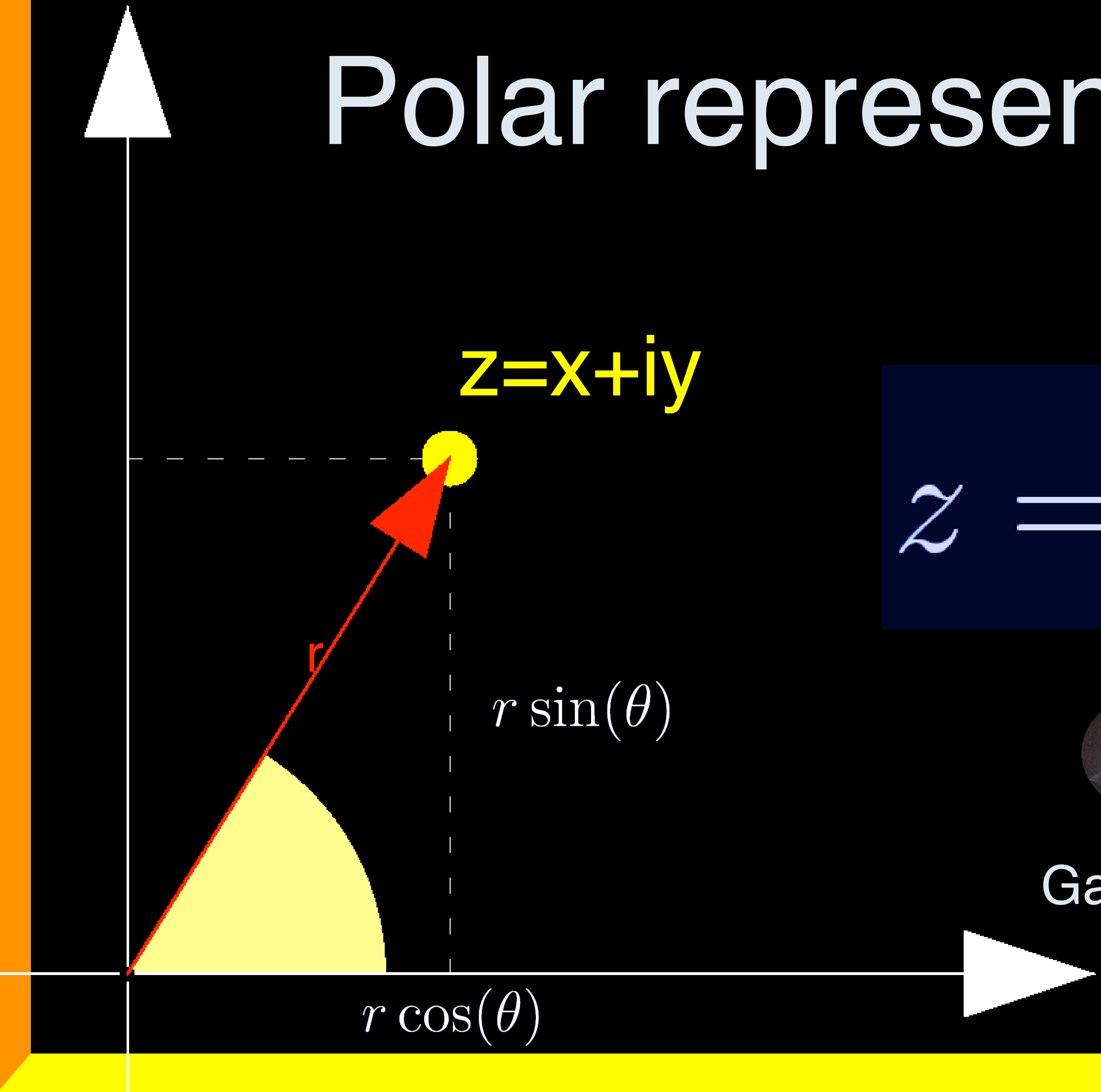
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

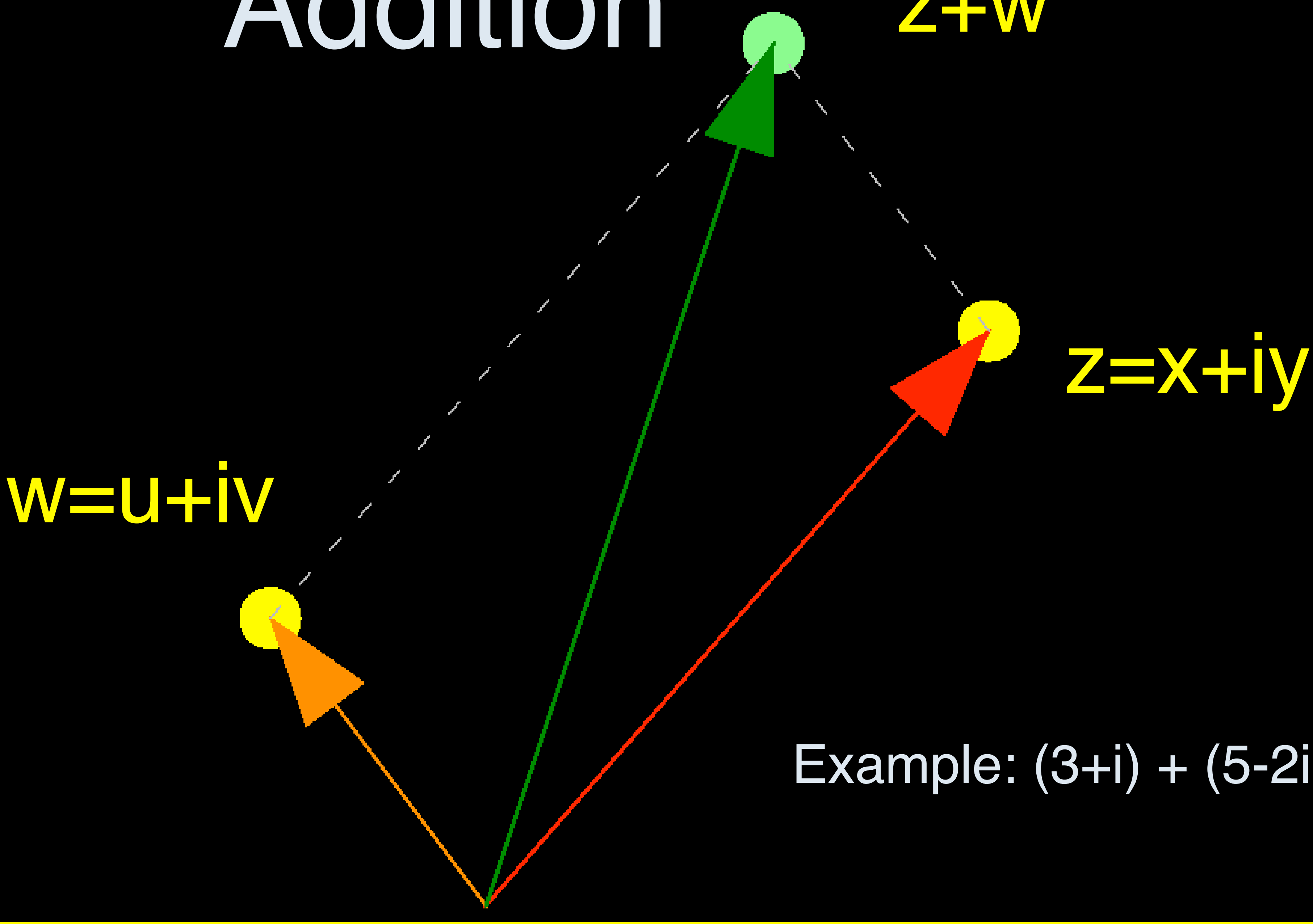
Polar representation



$$z = re^{i\theta}$$

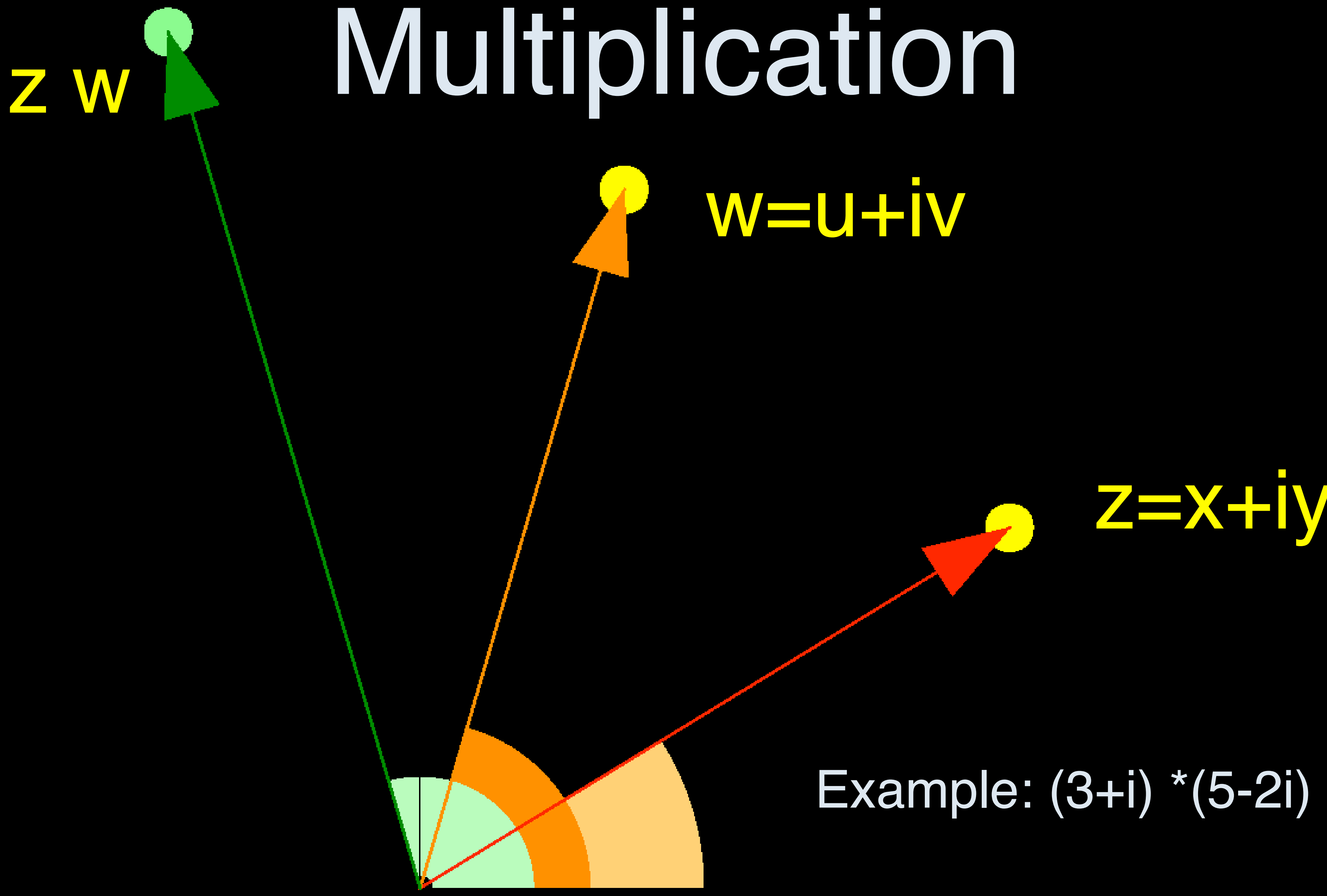
Gauss Plane

Addition



Example: (3+i) + (5-2i)

Multiplication

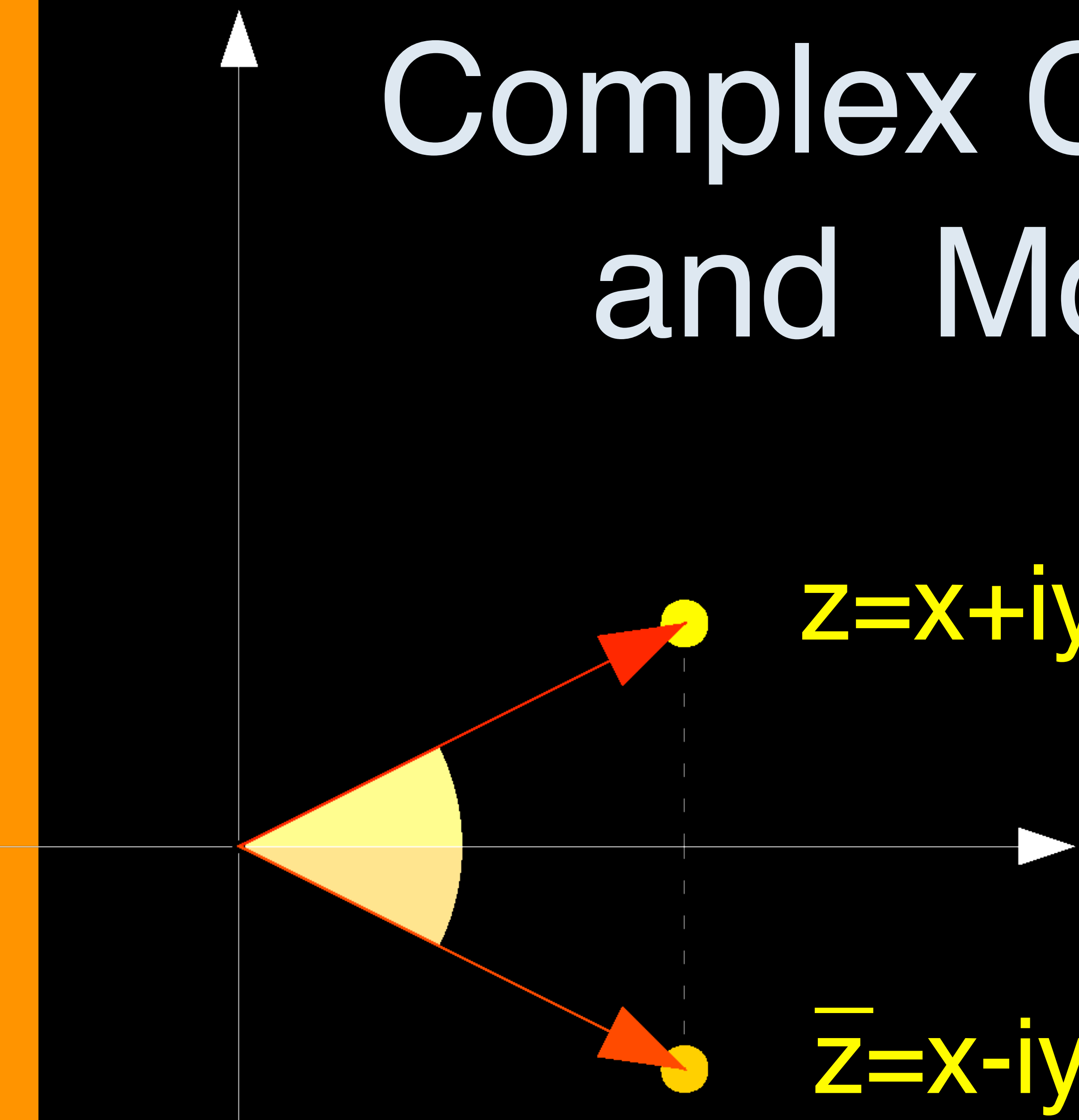


Example: (3+i) *(5-2i)

The most remarkable formula in math

$$1 + e^{i\pi} = 0$$

Complex Conjugate and Modulus

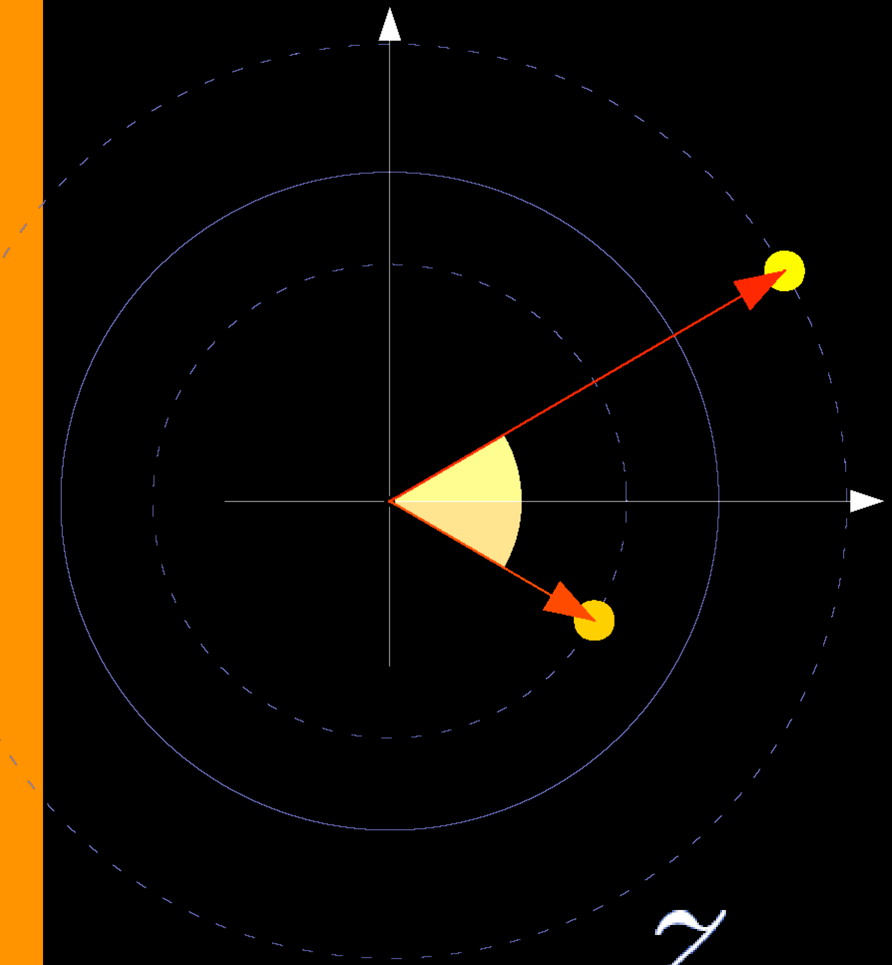


$z = x + iy$

$|z|^2 = z\bar{z}$

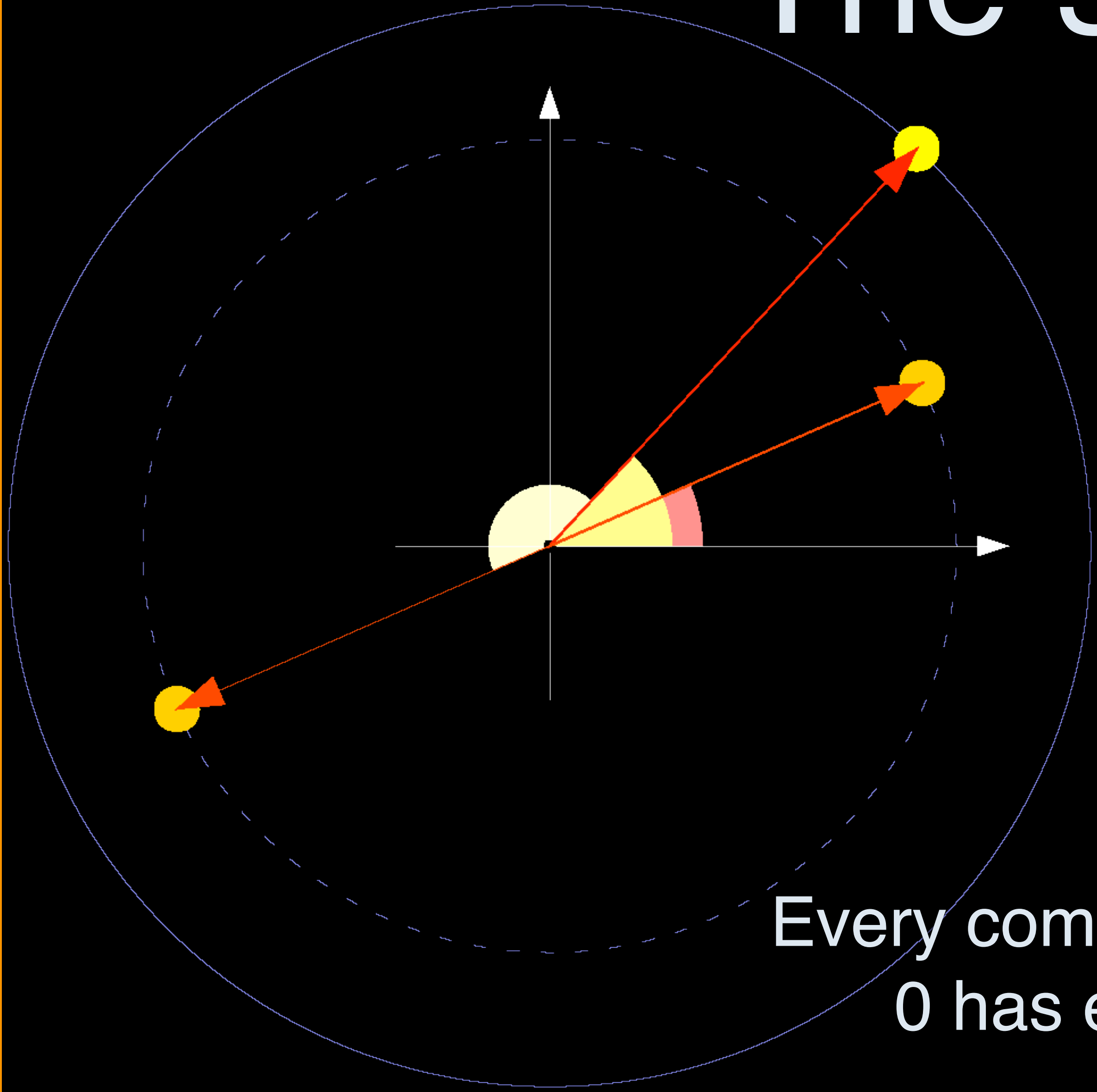
$\bar{z} = x - iy$

Division



$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

The square root

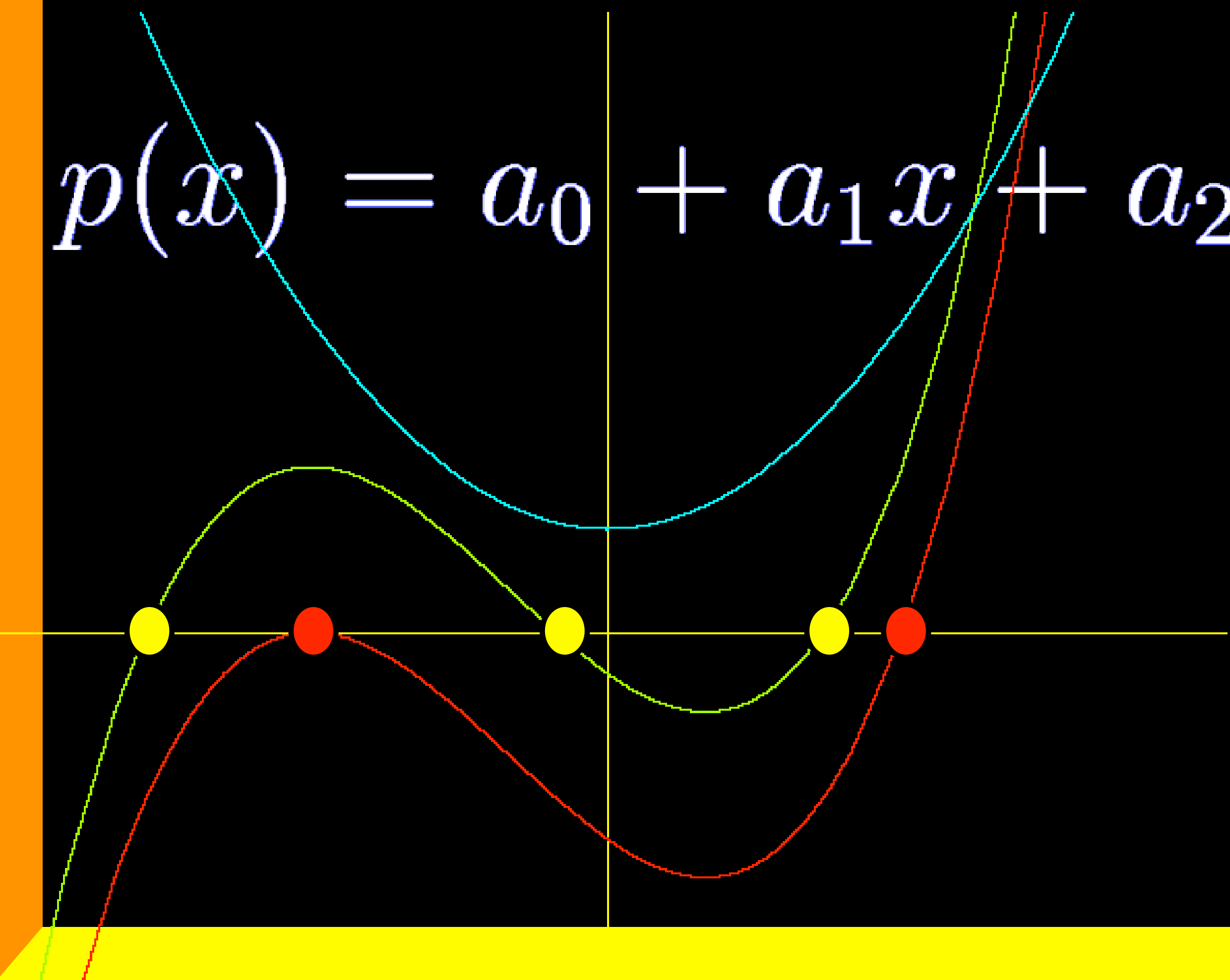


"Take square root of modulus and divide angle by 2".

Every complex number different from 0 has exactly 2 square roots.

Fundamental theorem of algebra

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$



A polynomial of degree n has exactly n roots p(x) =0

Eigenvalues

$$A\,v = \lambda v$$

Examples:

A	v is in the kernel
$Q^T Q = I$	v is in the rotation axes
$Q^2 = I$	v is on the reflection space

Because det $(\lambda - A) = 0$, eigenvalues are the roots of the characteristic polynomial

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f(\lambda) = \lambda^2 - tr(A)\lambda + det(A)$$

$$\lambda_+ = \frac{tr(A) + \sqrt{tr(A)^2 - 4det(A)}}{2}$$

$$\lambda_- = \frac{tr(A) - \sqrt{tr(A)^2 - 4det(A)}}{2}$$

algebraic multiplicity

number of simultaneous roots

geometric multiplicity

dimension of ker(A-λI)

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$f_A(\lambda) = (\lambda - a)(\lambda - a)$$

algebraic multiplicity: 2
geometric multiplity: 1

Some good things to know

determinant

$$\det(A) = \lambda_1 \ \lambda_2 \ \cdots \ \lambda_n$$

trace

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

determinant of power

$$\det(A^k) = \lambda_1^k \lambda_2^k \cdots \lambda_n^k$$

trace of power

$$tr(A^k) = \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k$$

blackboard problem

Find the eigenvalues of

$$A = \begin{pmatrix} 6 & 5 & 5 & 5 & 5 & 5 \\ 5 & 6 & 5 & 5 & 5 & 5 \\ 5 & 5 & 6 & 5 & 5 & 5 \\ 5 & 5 & 5 & 6 & 5 & 5 \\ 5 & 5 & 5 & 5 & 6 & 5 \\ 5 & 5 & 5 & 5 & 5 & 6 \end{pmatrix}$$

Eigenvectors

$Av=\lambda v$

$(A-\lambda I)v=0$

v is in the kernel of $A-\lambda I$

The eigenspace is a kernel. If all eigenvalues are different, we have one eigenvector for each eigenvalue.

Example:

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\lambda_+ = a+ib \qquad v_+ = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_- = a-ib \qquad v_- = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

All you have to do to find the eigenvalues is

compute kernels!

blackboard problem

Find all the eigenvalues and eigenvectors of

$$A=\begin{pmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 5 \end{pmatrix}$$

Discrete dynamical systems

$$x(t+1) = A\,x(t)$$

has solution:

$$x(n) = A^{n}x(0)$$

but this does not give any insight.

$$x(0) = a_1 v_1 + \dots + a_n v_n$$

$$A v_k = \lambda_k v_k$$

$$x(t) = a_1 \lambda_1^t v_1 + \dots + a_n \lambda_n^t v_n$$

Example: Fibonnacci

$$x(n+1) = x(n) + x(n-1) \qquad 1,1,2,3,5,8,13,21,\ldots$$

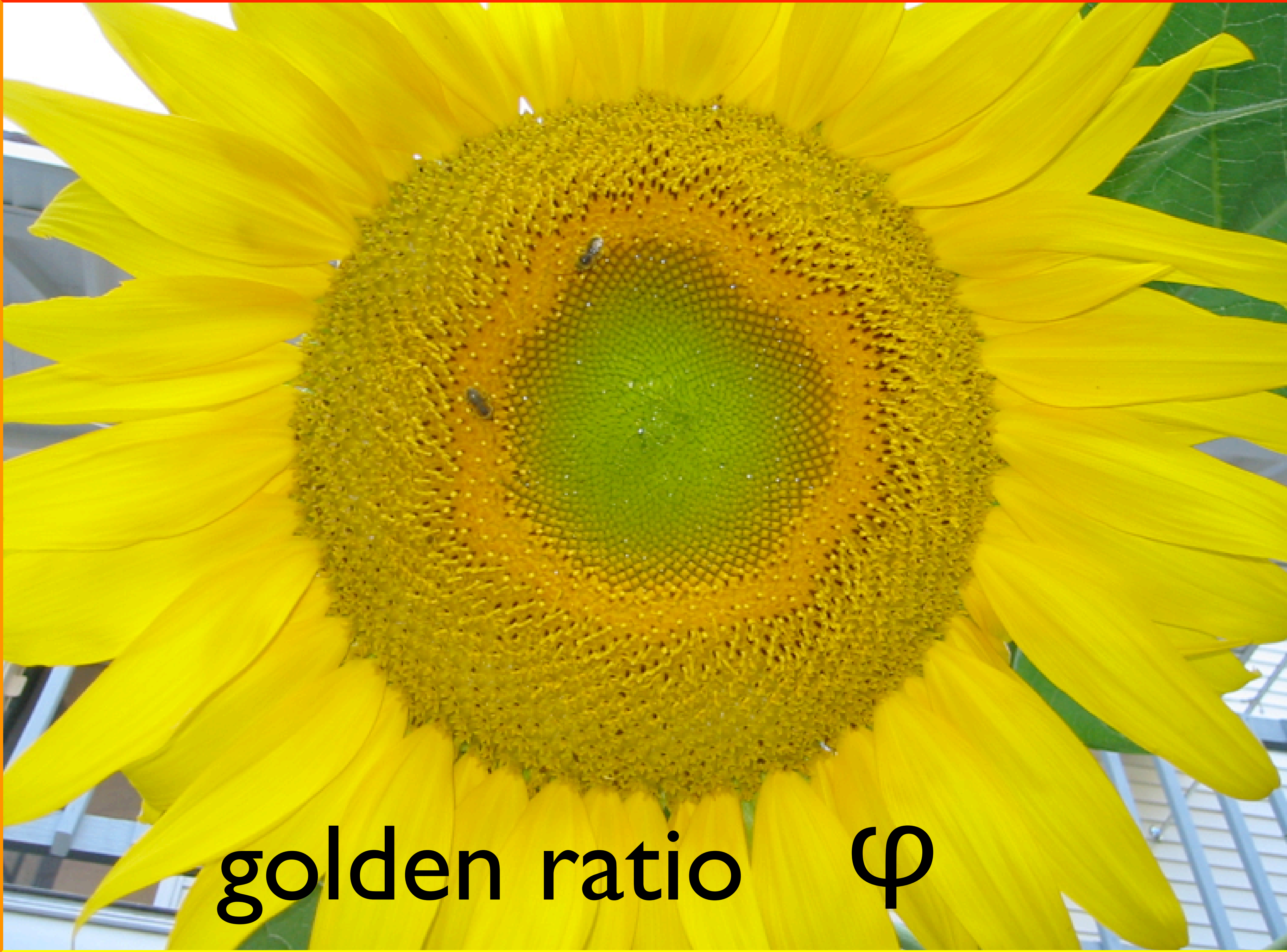
$$\begin{pmatrix} x(n+1) \\ x(n) \end{pmatrix} = \begin{pmatrix} x(n)+x(n-1) \\ x(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(n) \\ x(n-1) \end{pmatrix}$$

characteristic polynomial: $\lambda^2 - \lambda - 1 = f_A(\lambda)$

eigenvalues $=\varphi, 1/\varphi$ eigenvectors $= \begin{pmatrix} \varphi \\ 1 \end{pmatrix} \begin{pmatrix} 1/\varphi \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\varphi \\ 1 \end{pmatrix}$$

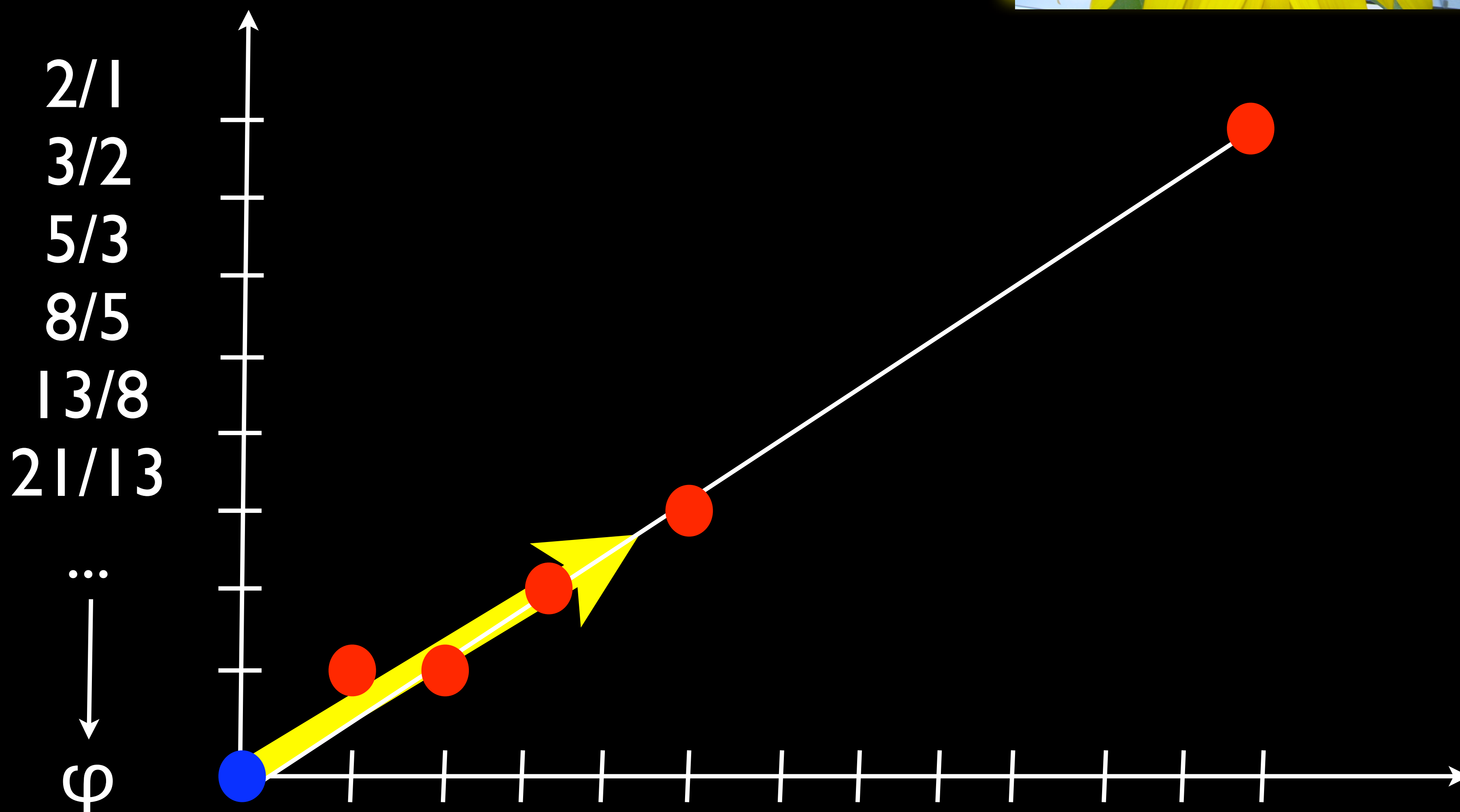
$$x(n) = (\varphi^n + 1/\varphi^n)/\sqrt{5}$$


golden ratio φ

golden ratio

$$\varphi = \frac{\sqrt{5}+1}{2} = 1.616...$$

$$1/\varphi=\varphi-1=0.616...$$



blackboard problem

The Lilac Bush problem

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

new branches
old branches

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$

Questions?