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- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1) [T] [F]   Let A and B be two $n \times n$ matrices. Then A and B are similar if and only if they have the same characteristic polynomials.

2) [T] [F]   Let A be a $n \times n$ matrix. Applying Gauss-Jordan row operations to A does never change the determinant.

3) [T] [F]   An orthogonal matrix is symmetric $A^T = A$ or anti symmetric $A^T = -A$.

4) [T] [F]   The eigenvalues of a matrix A do not change under row reduction.

5) [T] [F]   The eigenvectors of a matrix A do not change under row reduction.

6) [T] [F]   If A is the matrix of a reflection, then $\det(A + I) = 0$.

7) [T] [F]   Every upper triangular matrix can be diagonalized.

8) [T] [F]   There is a recursion $x_{n+1} = a x_n + b x_{n-1}$ for which $x_n > (1.01)^{2^n}$ for all n.

9) [T] [F]   The sum of two projections is a projection.

10) [T] [F]   The characteristic polynomial of A is the same as the characteristic polynomial of A^T.

11) [T] [F]   For any matrix, we have $\det(A^5) = \det(A)^5$.

12) [T] [F]   The trace of a matrix is equal to the product of the eigenvalues.

13) [T] [F]   There is a projection for which the determinant is equal to 2.

14) [T] [F]   The matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$.

15) [T] [F]   If two 2×2 matrices A and B have the same trace and determinant, then they are similar.

16) [T] [F]   If x is a solution to the linear equation $Ax = b$, then x is a least square solution to $Ax = b$.

17) [T] [F]   It is possible that the length of $A^n v$ and the length of $A^{-n} v$ both grow exponentially.

18) [T] [F]   If an orthogonal matrix Q is symmetric, then Q is diagonal.

19) [T] [F]   If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.

20) [T] [F]   For every invertible $n \times n$ matrix A, there is a nonzero $n \times n$ matrix B such that AB is the zero matrix.

[   ] Total

Which matrices are orthogonal, which matrices are symmetric, which matrices are projections? No explanations are necessary for this problem. Check everything which applies. It is not excluded that you have to check several properties for each matrix.

	orthogonal	symmetric	projection	
a)				$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
b)				$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
c)				$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
d)				$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Which of the following assertions is true?

true	false	
		A is similar to B
		A is similar to C
		A is similar to D
		B is similar to C
		B is similar to D
		C is similar to D

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A with their algebraic multiplicities.

b) Find the geometric multiplicities of each of the eigenvalues.

c) Find all the eigenvectors.

d) What is $\det(A)$?

Problem 4) (10 points)

Find the function $y = f(x) = a\cos(\pi x) + b\sin(\pi x)$, which best fits the data

x	y
0	1
1/2	3
1	7

Problem 5) (10 points)

a) Find all the eigenvalues λ_1, λ_2 and λ_3 of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Find a formula for $\text{tr}(A^n)$.

Problem 6) (10 points)

a) (5 points) Find an eigenbasis of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

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b) (5 points) Do Gram-Schmidt orthogonalization on the basis $\mathcal{B} = \{v_1, v_2, v_3 \}$ you just got. Write down the QR decomposition of the matrix S which contains the basis $\mathcal{B}$ as column vectors.

Problem 7) (10 points)

a) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 4 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

b) (5 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}$$

Problem 8) (10 points)

We want to find a formula for the general term x_n in the recursion

$$x_{n+1} = x_n + 3x_{n-1}/4$$

if $x_0 = 0, x_1 = 1$. This is the case of a Fibonacci recursion, in which only $3/4$ of the previous generation has kids.

a) Write the recursion in the form $v_{n+1} = Av_n$ for vectors $v_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in the plane.

b) Find the eigenvalues λ_+, λ_- and eigenvectors v_+, v_- of A.

c) Write $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $v_0 = av_+ + bv_-$.

d) Find $v_n = A^n v_0$ and so x_n.

Our goal is to find the determinant of

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 \\ 2 & 102 & 2 & 2 & 2 & 2 \\ 3 & 3 & 103 & 3 & 3 & 3 \\ 4 & 4 & 4 & 104 & 4 & 4 \\ 5 & 5 & 5 & 5 & 105 & 5 \\ 6 & 6 & 6 & 6 & 6 & 106 \end{bmatrix}$$

a) The matrix $A - 100 I_6$ has an eigenvalue 0. Find its algebraic multiplicity.

b) The matrix A^T has an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the corresponding eigenvalue λ.

c) Why does A also have the same eigenvalue λ?
d) You have found all the eigenvalues of $A - 100 I_6$. What are the eigenvalues of A?
e) Find the determinant of A.