ORTHOGONAL MATRICES
Math 21b, O. Knill

Section 5.3: 5-8,9-11,13-16,17-20,40,48*,44*

TRANSPOSE The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

PROPERTIES.

a) $(AB)^T = B^T A^T$.

b) $\vec{x} \cdot A\vec{y} = A^T \vec{x} \cdot \vec{y}$.

c) $(A^T)^T = A$.

PROOFS.

a) $(AB)_{kl} = \sum_i A_{ki} B_{il}$. $(AB)^T_{kl} = \sum_i A_{li} B_{ik} = A^T B^T$.

b) $\vec{x} \cdot A\vec{y}$ is the a matrix product $x^T A y$ Use $(x^T A) = (A^T x)^T$ from a) to see $x^T A y = (A^T x)^T y = A^T x \cdot y$.

c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.

ORTHOGONAL MATRIX. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The corresponding linear transformation is called **orthogonal**.

INVERSE. It is easy to invert an orthogonal matrix: $A^{-1} = A^T$.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix A have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

PRESERVATION OF LENGTH AND ANGLE. Orthogonal transformations preserve the dot product:
$A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ Proof. $A\vec{x} \cdot A\vec{y} = A^T A\vec{x} \cdot \vec{y}$ and because of the orthogonality property, this is $\vec{x} \cdot \vec{y}$.

> Orthogonal transformations preserve the **length** of vectors as well as the **angles** between them.

Proof. We have $||A\vec{x}||^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x} \, ||\vec{x}||^2$. Let α be the angle between $\vec{x}$ and $\vec{y}$ and let β denote the angle between $A\vec{x}$ and $A\vec{y}$ and α the angle between $\vec{x}$ and $\vec{y}$. Using $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ we get $||A\vec{x}|| ||A\vec{y}|| \cos(\beta) = A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y} = ||\vec{x}|| ||\vec{y}|| \cos(\alpha)$. Because $||A\vec{x}|| = ||\vec{x}||, ||A\vec{y}|| = ||\vec{y}||$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate $\vec{x}$ in plane V spanned by $\vec{x}$ and $\vec{y}$ by an angle ϕ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all ϕ. Differentiation with respect to ϕ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis. Proof. Look at $A^T A = 1_n$. Each entry is a dot product of a column of A with an other column of A.

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1_n$.

EXAMPLES.
The composition of two reflections at a line is a rotation.
The composition of two rotations is a rotation.
The composition of a reflections at a plane with a reflection at an other plane is a rotation (the axis of rotation is the intersection of the planes).

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ is the matrix $\boxed{AA^T}$, where A is the matrix with column vectors $\vec{v}_i$. To see this just translate the formula $P\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \ldots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$ into the language of matrices: $A^T \vec{x}$ is a vector with components $\vec{b}_i = (\vec{v}_i \cdot \vec{x})$ and $A\vec{b}$ is the sum of the $\vec{b}_i \vec{v}_i$, where $\vec{v}_i$ are the column vectors of A. Orthogonal projections are no orthogonal transformations unless it is the identity!

EXAMPLE. Find the orthogonal projection P from $\mathbf{R}^3$ to the linear space spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \frac{1}{5}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Solution: $AA^T = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{bmatrix}$.

WHY ARE ORTHOGONAL TRANSFORMATIONS USEFUL?

- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.

- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.

- In the QR decomposition of a matrix A, the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.

- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (like the JPG image format) and sound compression (like the MP3 sound format).

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

Yes	No	Shear in the plane.
Yes	No	Projection in three dimensions onto a plane.
Yes	No	Reflection in two dimensions at the origin.
Yes	No	Reflection in three dimensions at a plane.
Yes	No	Dilation with factor 2.
Yes	No	The Lorenz boost $\vec{x} \mapsto A\vec{x}$ in the plane with $A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$
Yes	No	A translation.

CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotate the point $(0,0)$ to (a,b) a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2), we form $A = R_{a_2,b_2} R_{a_1,b_1}^{-1}$.

EXAMPLE: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix

$A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix}$.