Math21b

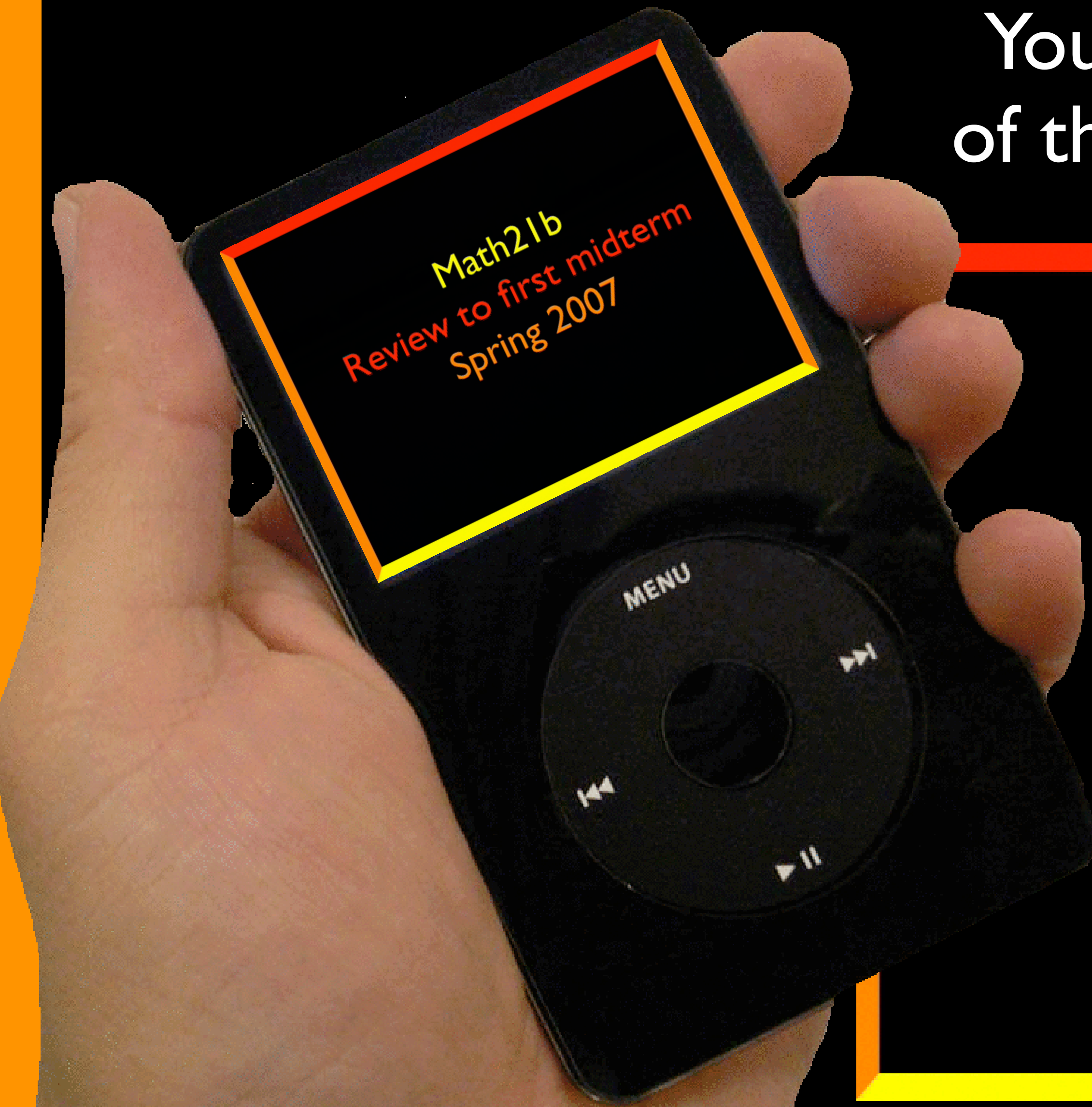
Review to final

Spring 2007

Oliver Knill, Mai 14, 2007

We focus on the material after the 2. midterm, but repeat now some key points

You can go through the slides of the 2 previous reviews again.



Plan for today:

- Key points from before second midterm
- Discrete dynamical systems
- Linear differential equations
- Nonlinear systems
- Operator methods
- Fourier theory
- Partial differential equations

Key points from before second midterm

$A x = b$

$x = A^{-1} b$

row reduce
$[A|\, b]$

Linear
equations

Solutions
are $x+\ker(A)$

Least square
solution

consistent:
have solution

chalk problem

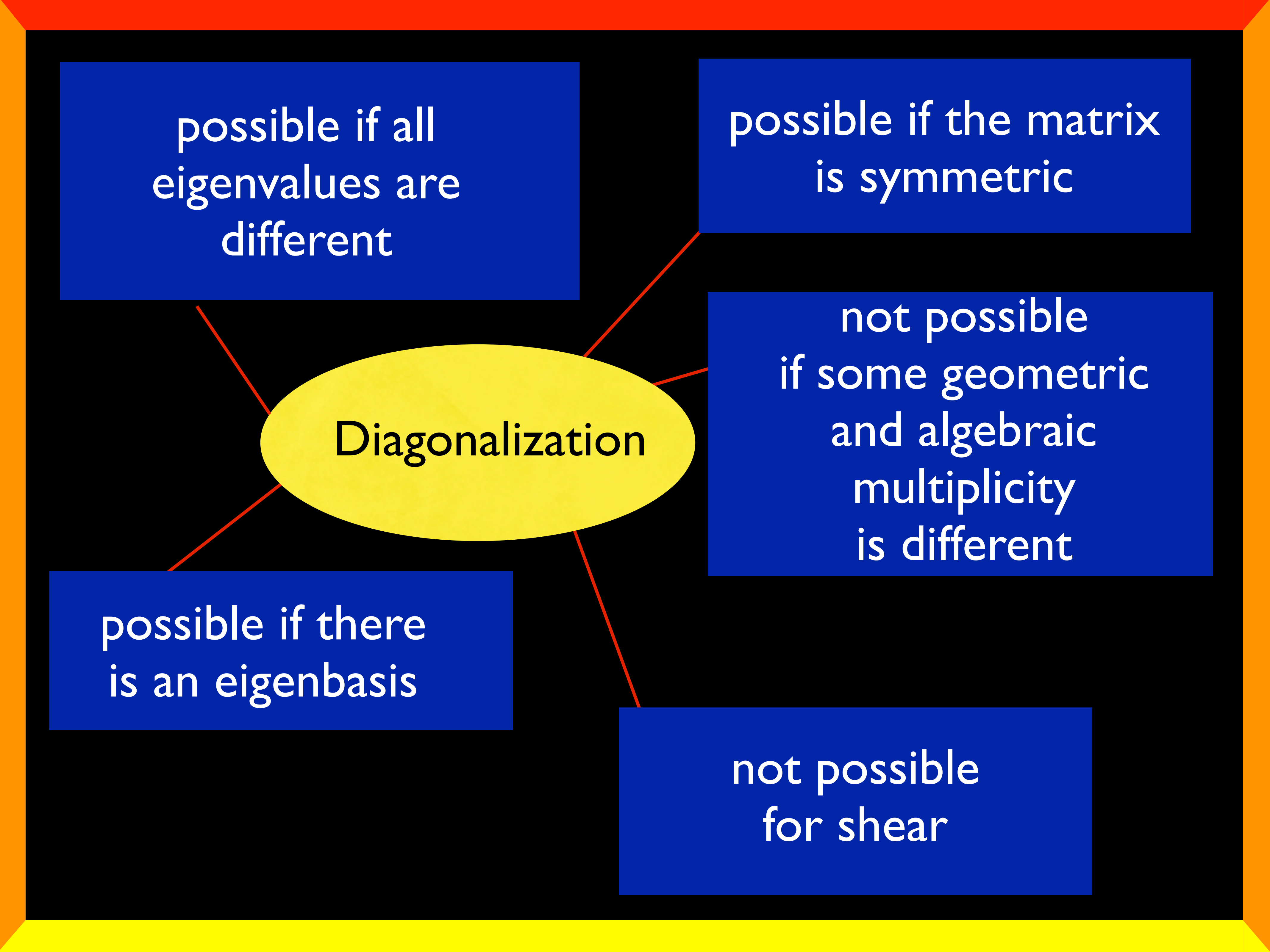
general solution of:

$$x+y+z = 6$$

$$2x-y+5z = 15$$

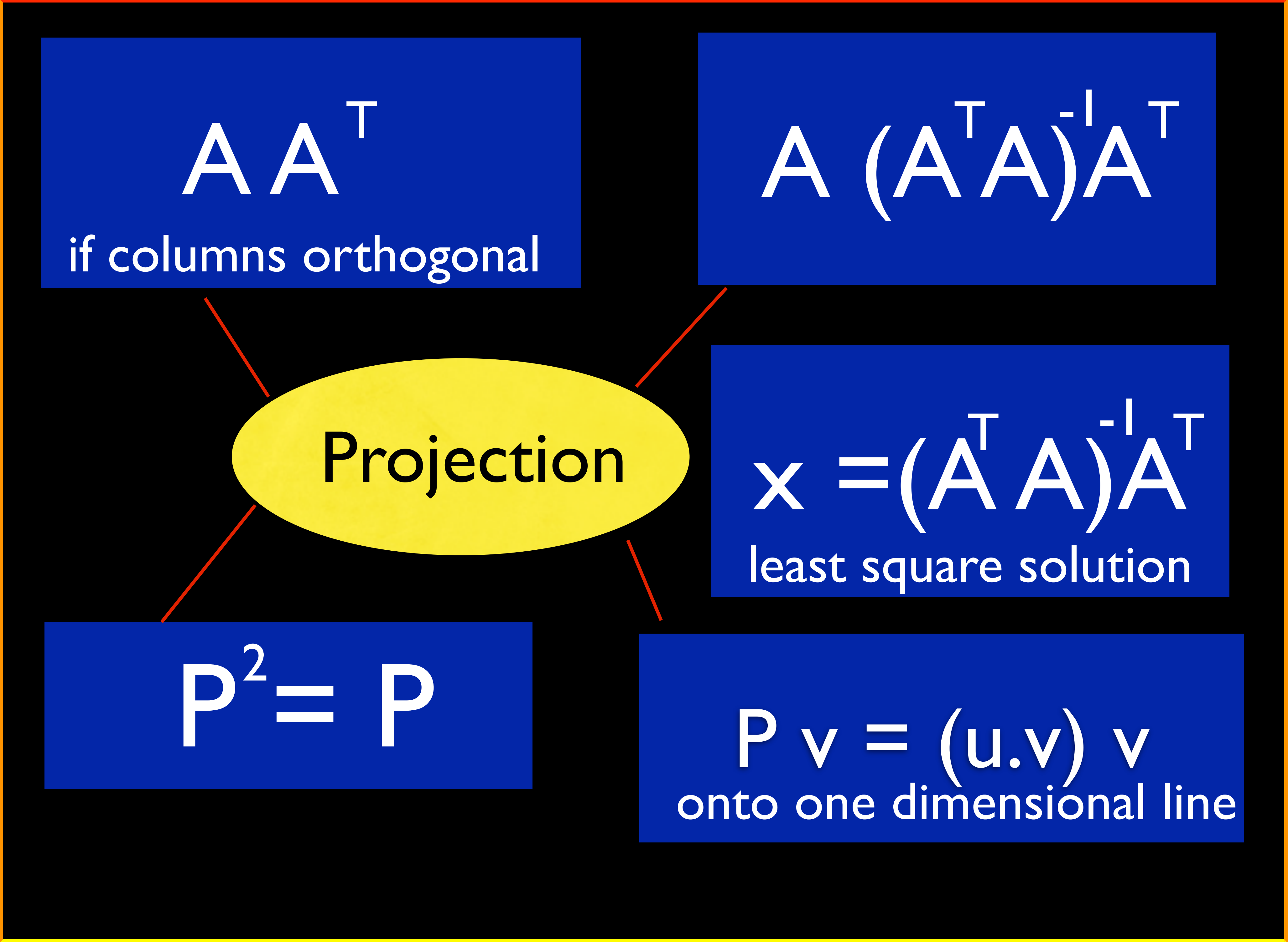
$$4x+y+7z = 27$$

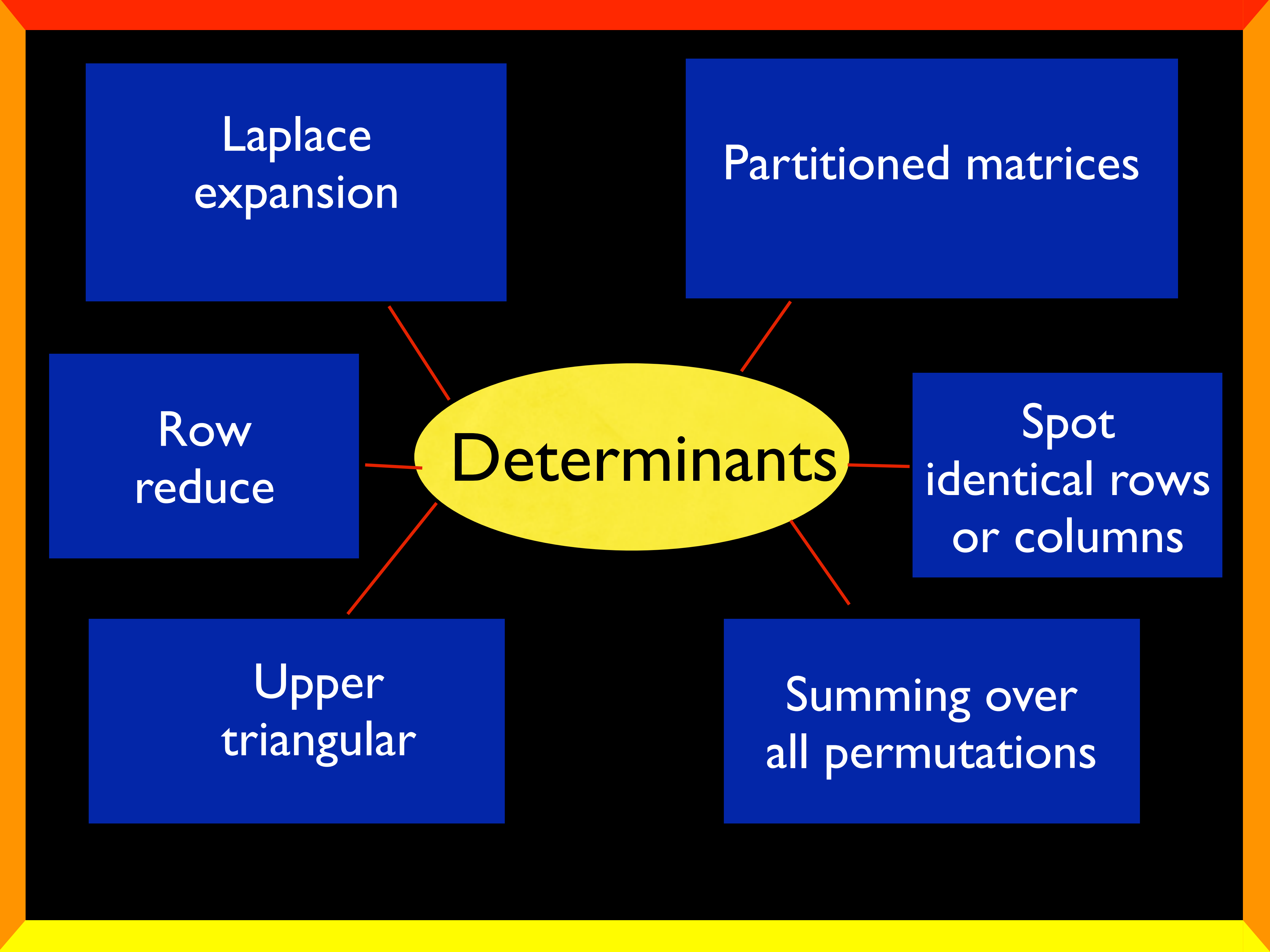
Row reduce the augmented matrix and write solution as sum of special solution and kernel parametrized by free variables.


possible if all eigenvalues are different
possible if the matrix is symmetric
Diagonalization
not possible if some geometric and algebraic multiplicity is different
possible if there is an eigenbasis
not possible for shear

Can we diagonalize?

1	2	3	0	0	0	0
0	8	2	0	0	0	0
0	0	9	0	0	0	0
0	0	0	1	0	0	0
0	0	0	4	6	0	0
0	0	0	9	10	11	0
0	0	0	13	14	15	16


$A A^T$
if columns orthogonal
$A (A^T A)^{-1} A^T$
Projection
$x = (A^T A)^{-1} A^T$
least square solution
$P^2 = P$
$P v = (u.v) v$
onto one dimensional line


Laplace expansion
Partitioned matrices
Row reduce
Determinants
Spot identical rows or columns
Upper triangular
Summing over all permutations

no chalk problem

$$\det\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 7 & 8 & 9 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 & 15 & 16 \end{pmatrix} = ?$$

This example was motivated from a story in the first issue of the Harvard student journal The Harvard College Mathematics Review

Endpaper: How to Compute Determinants

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During one of my years in graduate school in Israel, I was a teaching fellow for a class on linear algebra. I found the job annoying for two reasons: On one hand, the students were primarily non-math majors. But more importantly, my class started at eight in the morning, which did not rhyme well with my lifestyle at the time. As a result, I could not bring myself to prepare my section in advance. Instead I improvised each time....

One day I found myself explaining determinants. "You know, for a generic matrix a determinant is never zero. Somebody, give me an example of a matrix!" The class produced no reply. They were no less sleepy than I was. In fact, not only were they asleep but they were suspicious as well. They did not want to risk giving a matrix which by misfortune would have a zero determinant, with the gloomy title of "degenerate" attached to it.

So I proceeded: "OK, let's take the first matrix that comes to mind."

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

I set about computing the determinant by the usual formula. I was never good with computations and, once again, I was especially sleepy:

$$1 \cdot 5 \cdot 9 - 2 \cdot 4 \cdot 9 \pm 3 \cdot 4 \cdot 8 + \ldots .$$

It took me a good 10 minutes. And what a shock, the determinant was zero! "I must have made a mistake," I told the class. I ran through the calculations once more, checking every step. Another 10 minutes passed. Zero again!

I tried to save myself. "OK, but sometimes the determinant is zero. Sorry. But now let's take a *really* generic matrix."

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

Another lengthy computation. . . .

At the end of that semester I was forced to enroll in a special seminar for delinquent instructors.

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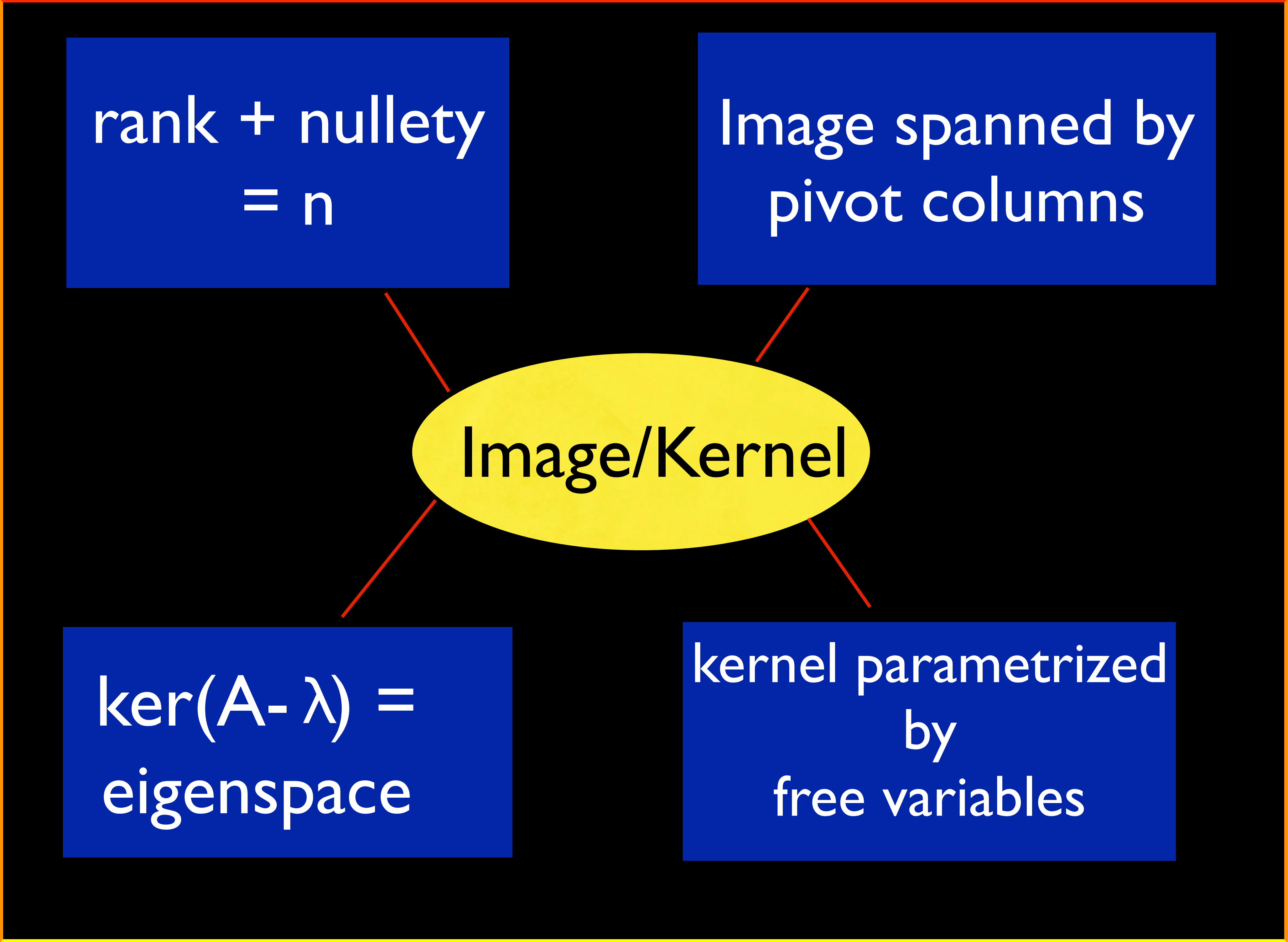
†Prof. Dennis Gaitsgory is a faculty member of the Harvard Mathematics Department.


rank + nullity
= n
Image spanned by
pivot columns
Image/Kernel
ker(A- λ) =
eigenspace
kernel parametrized
by
free variables

Matrices

n x m matrix

n rows and m columns

Columns: image of basis vectors

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{T} = B^{T}A^{T}$$

$$AB \neq BA$$

in general

$$B = S^{-1} A S$$

similarity

Complex Numbers

$z = a + i\,b$

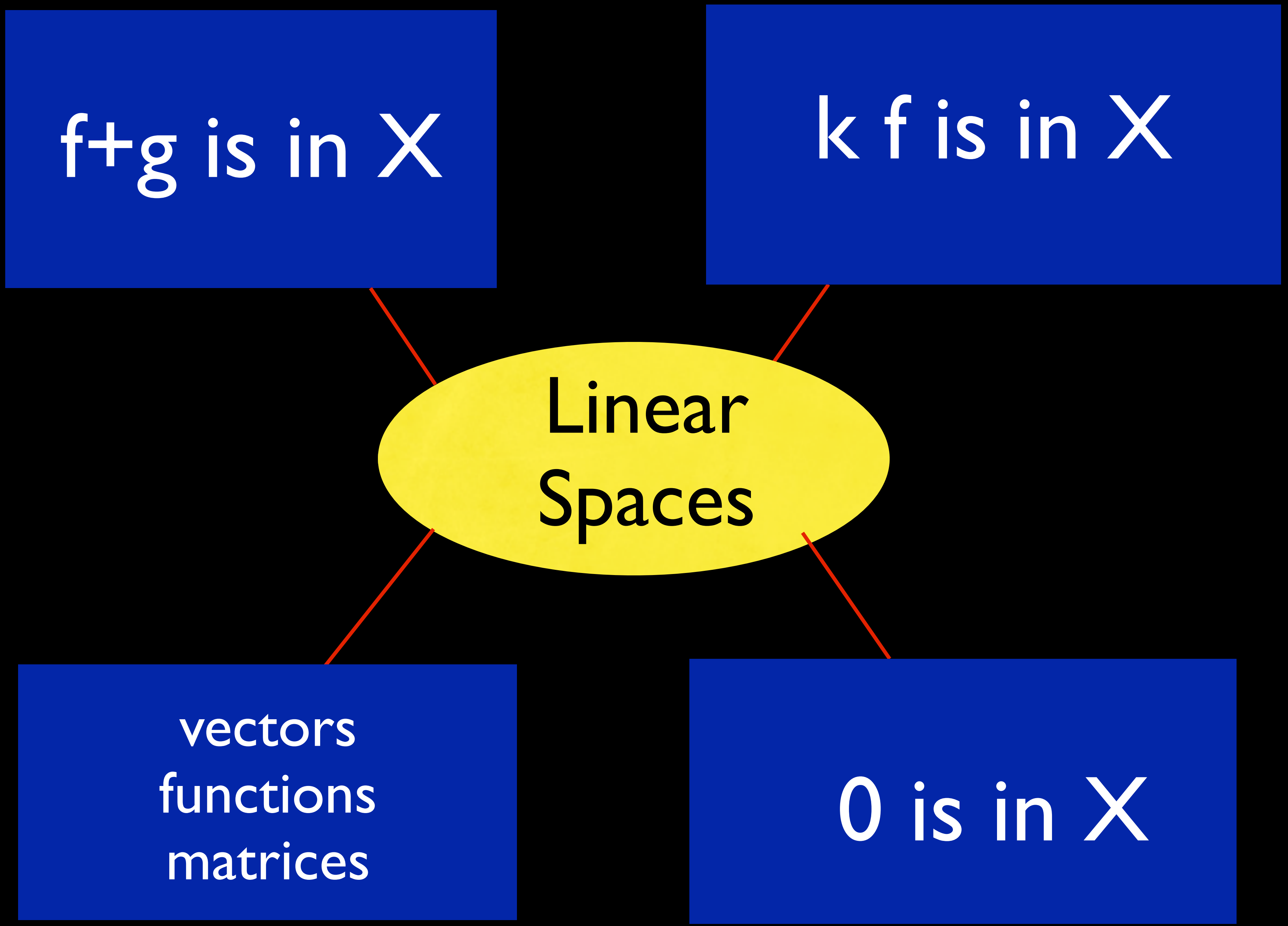
$z = r\cos(t) + i\,r\sin(t)$

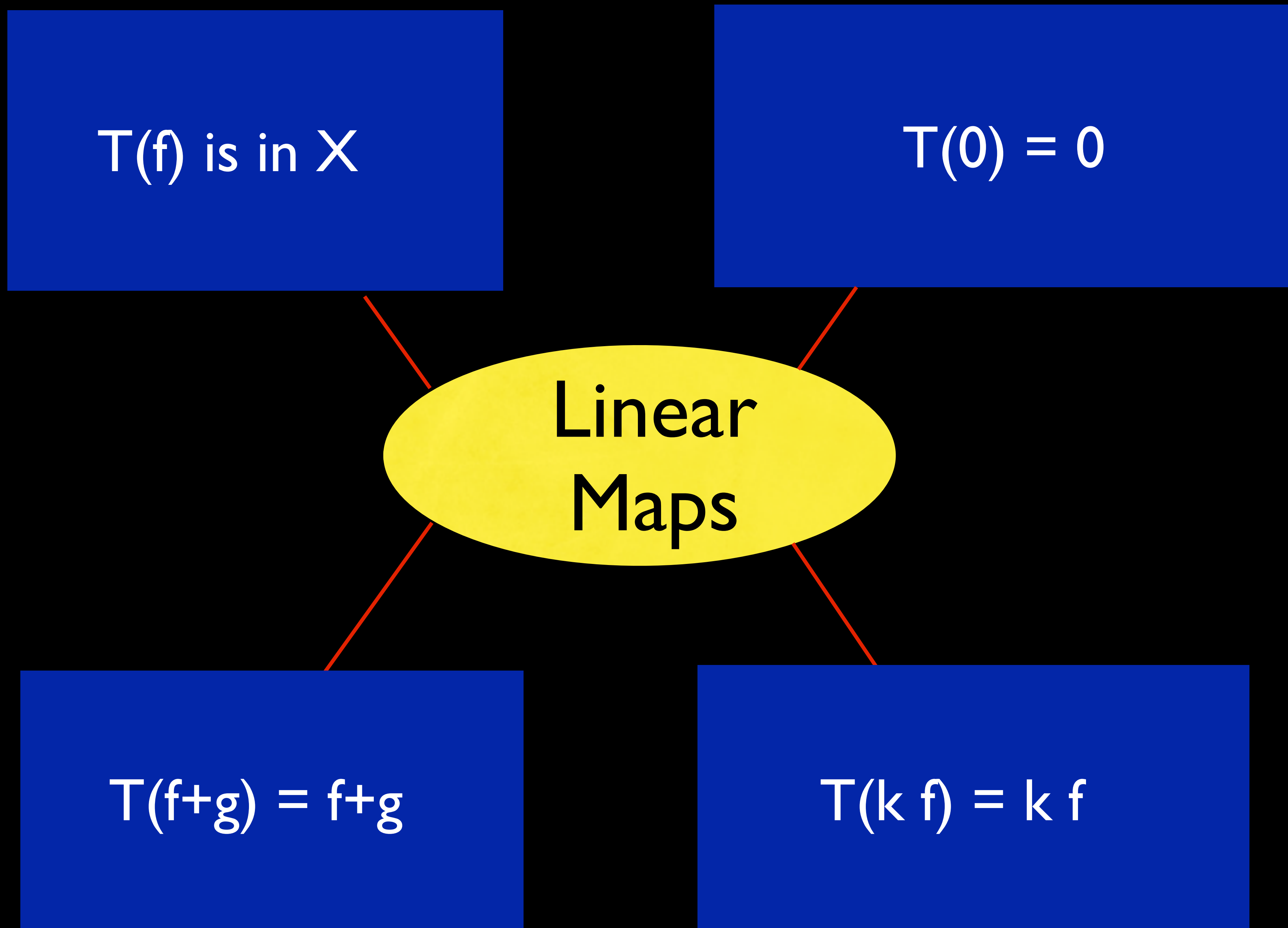
Euler formula

fundamental theorem of algebra

$f(\lambda) = \det(A-\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$

Happy birthday! Euler celebrated his 300 th birthday on April 15.


f+g is in X
k f is in X
Linear Spaces
vectors
functions
matrices
0 is in X


Linear Maps
T(f) is in X
T(0) = 0
T(f+g) = f+g
T(k f) = k f

no chalk problem

which of the following are linear spaces?

Smooth 2π-periodic functions with $\int_0^{2\pi} f(x)\,dx = 0$.

$\{ f \in C^\infty(\mathbf{R}) \mid f(10) = 1 \}$

Smooth 2π-periodic functions satisfying $f'(0) = 0$.

2×2 matrices satisfying $\mathrm{tr}(A) = 0$.

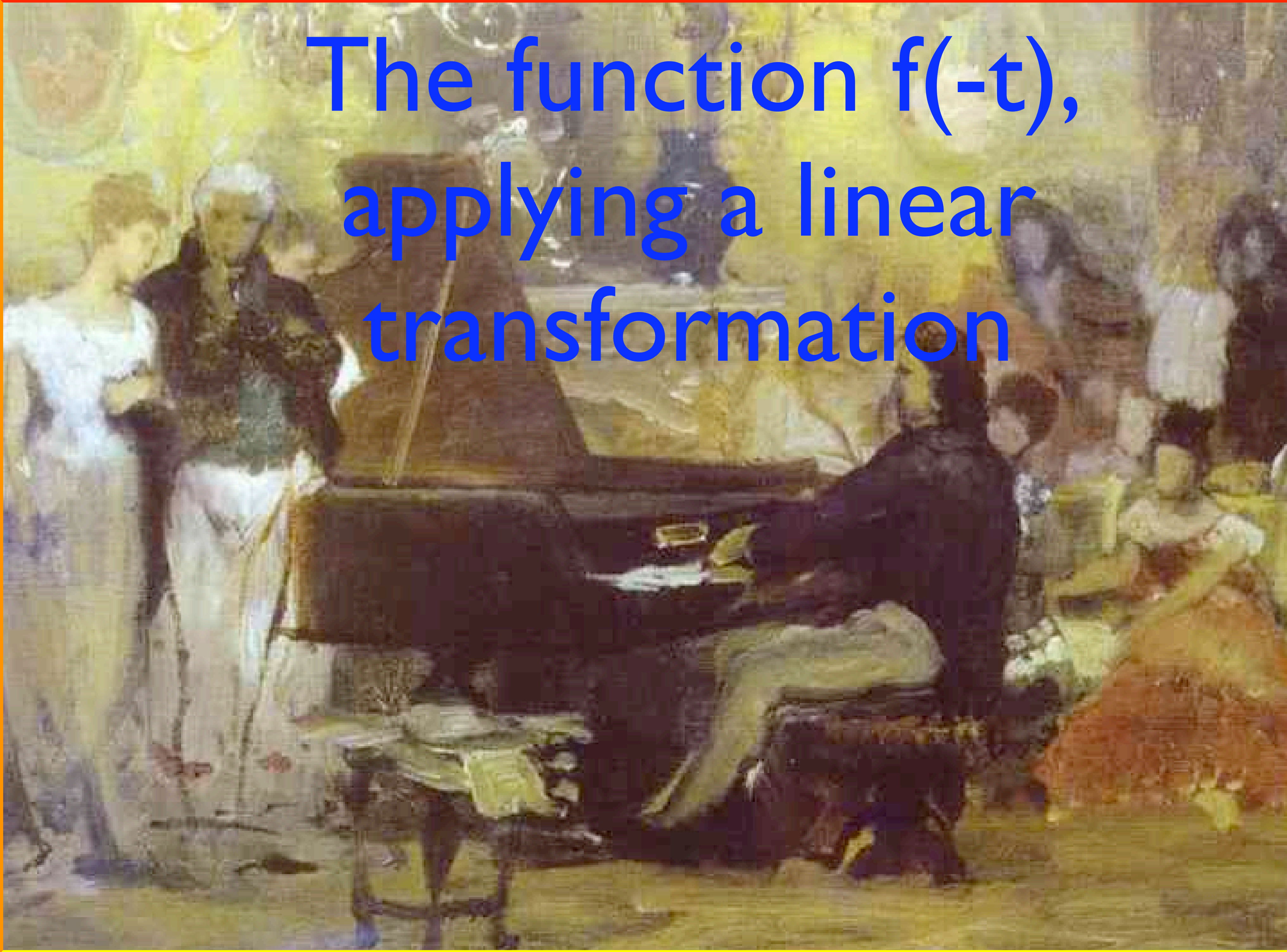
2×2 matrices satisfying $\det(A) = 0$.

$$T(f)(x) = x^2 f(x).$$
$$T(f)(x) = f(1)^2 + f(x).$$
$$T(f)(x) = f'(x).$$
$$T(f)(x) = f(x)f'(x).$$
$$T(f)(x) = x + f(x).$$

which T are linear transformations?

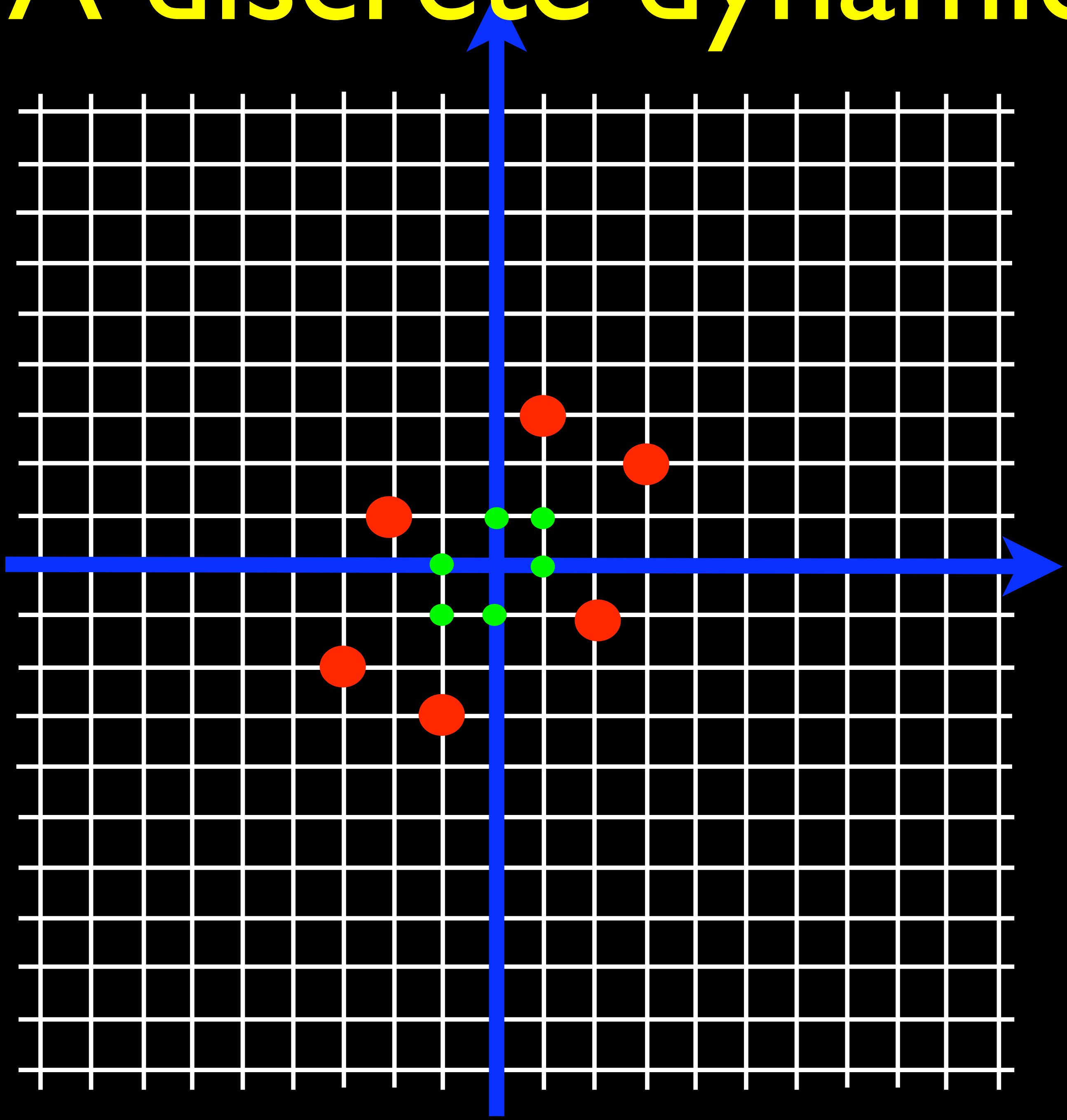
A music piece is just a function f(t)

The function f(-t), applying a linear transformation



Discrete dynamical systems

A discrete dynamical system



$$x(t+1) = x(t) - y(t)$$

$$y(t+1) = x(t)$$

1	-1
1	0

we have seen rabbits breed

$$x(t+1) = x(t) + x(t-1)$$

Lilac bushes grow


here is a story
about panda love:

blackboard problem

Problem

x(n) panda population at time n
y(n) panda actors in adult movies

$$x(n+1) = 2x(n) + y(n)$$
$$y(n+1) = x(n) + 2y(n)$$

Find a closed formula for x(n) and y(n)
if $x(0)=6$, $y(0)=2$

Discrete dynamical system

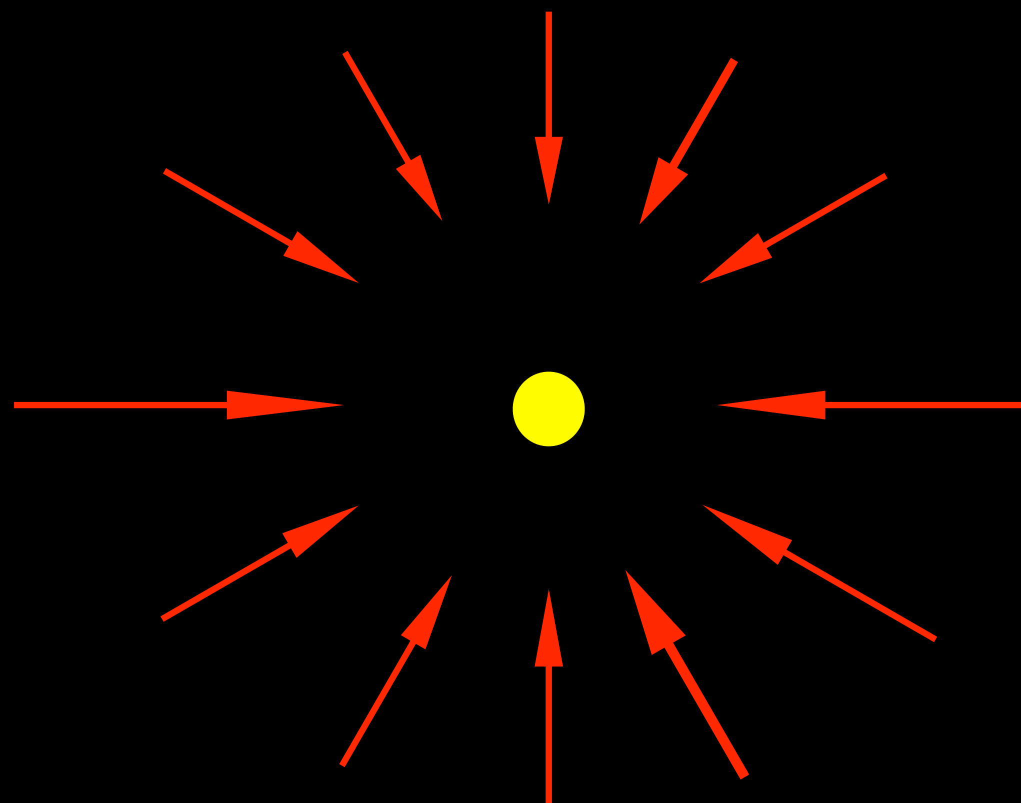
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

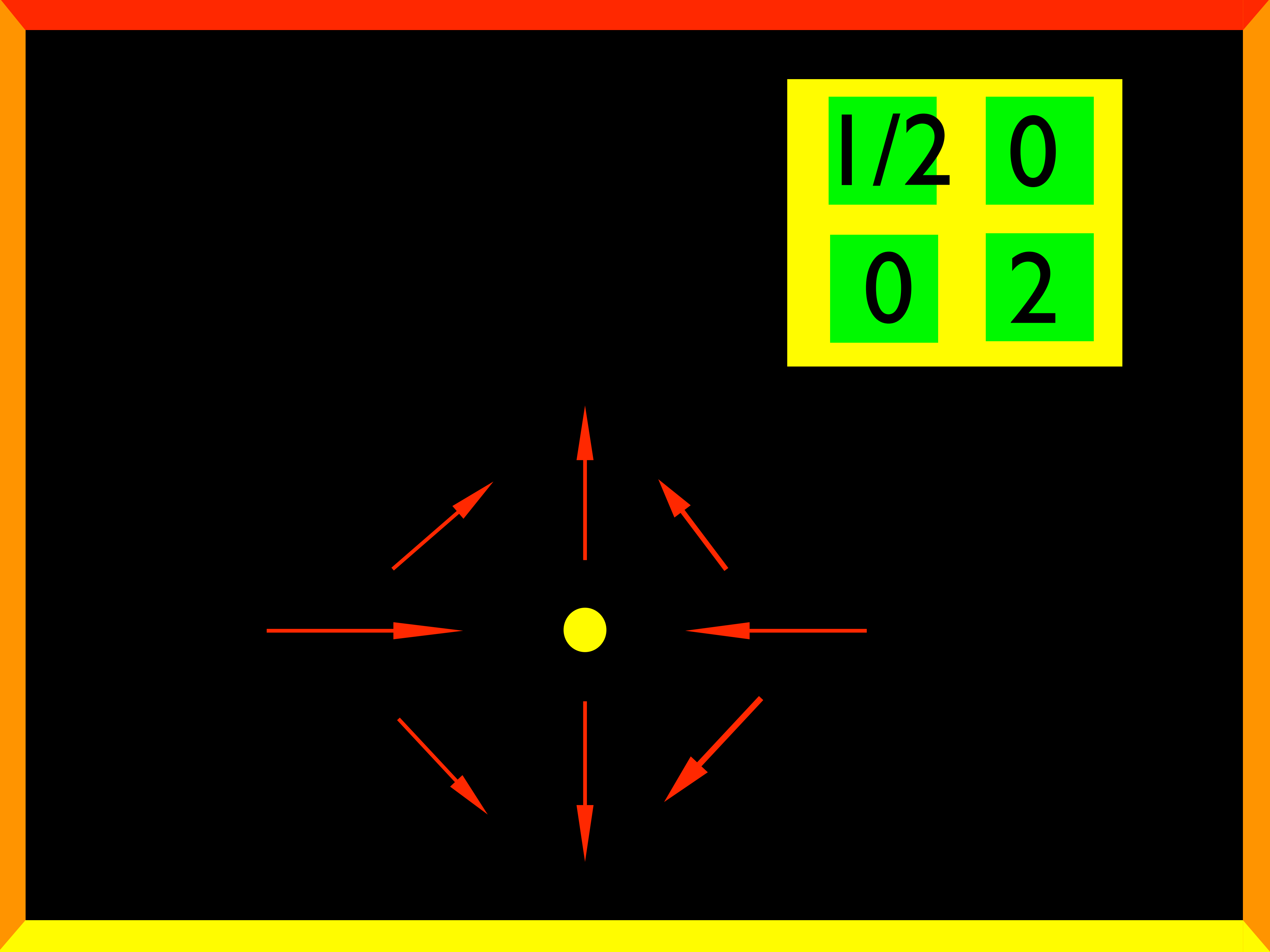
$$f(\lambda) = \lambda^2 - 2\lambda - 3$$

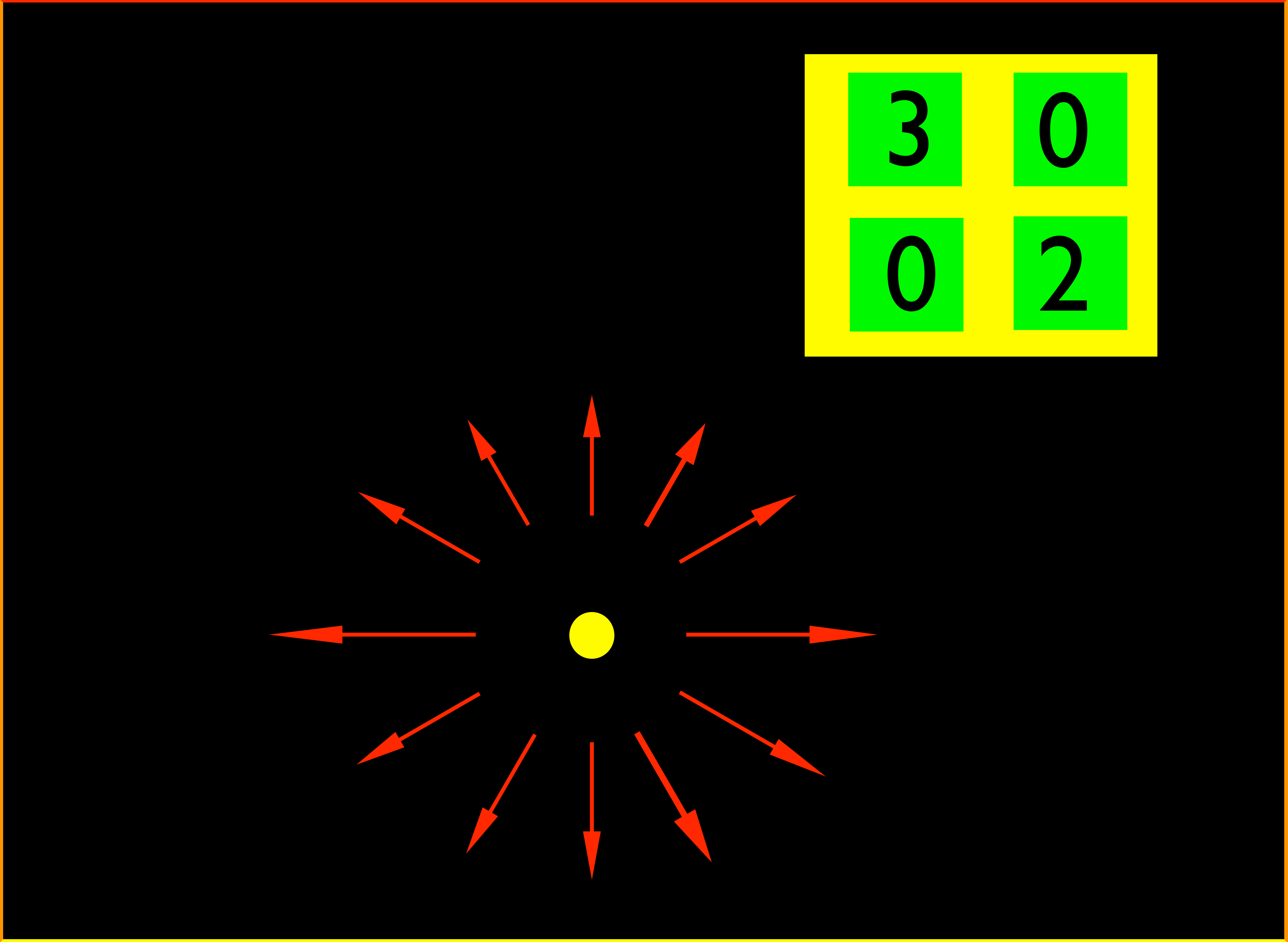
Eigenvalues: 3,-1 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Stability of discrete dynamical systems

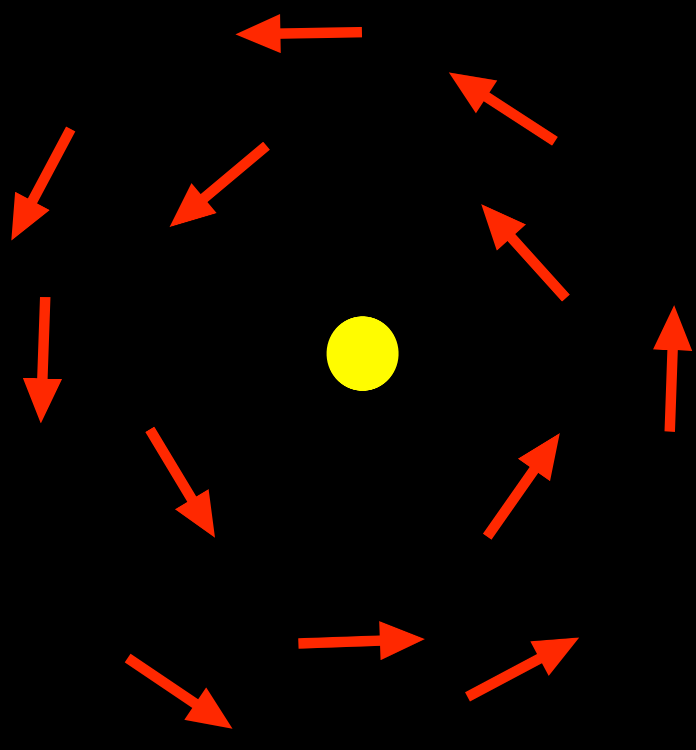
$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$$



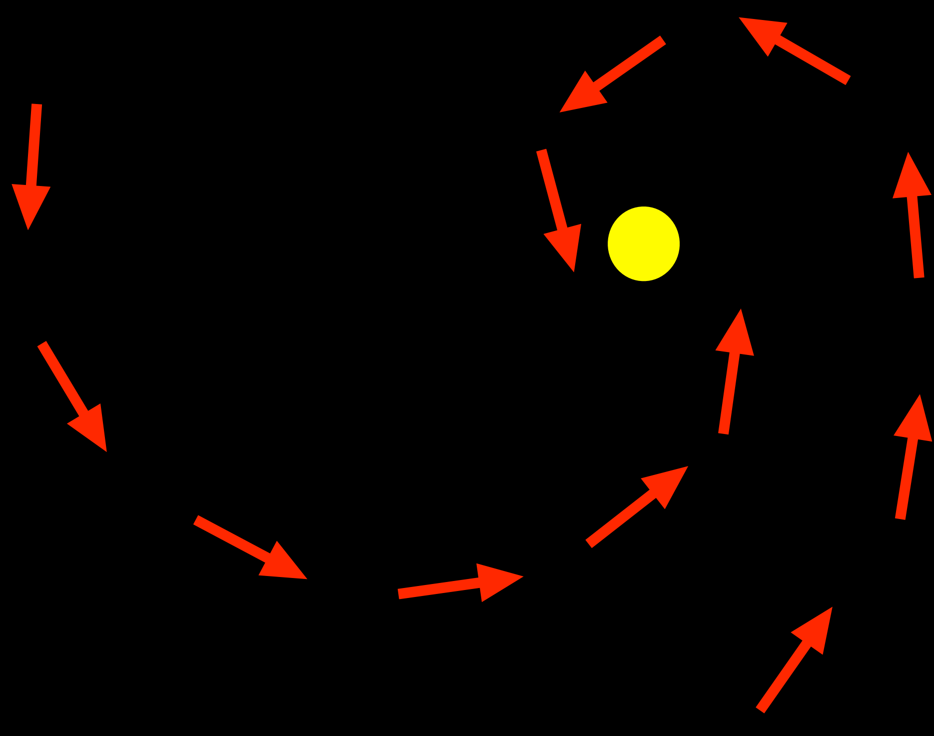

1/2 0
0 2


3
0
0
2

$$\begin{matrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{matrix}$$



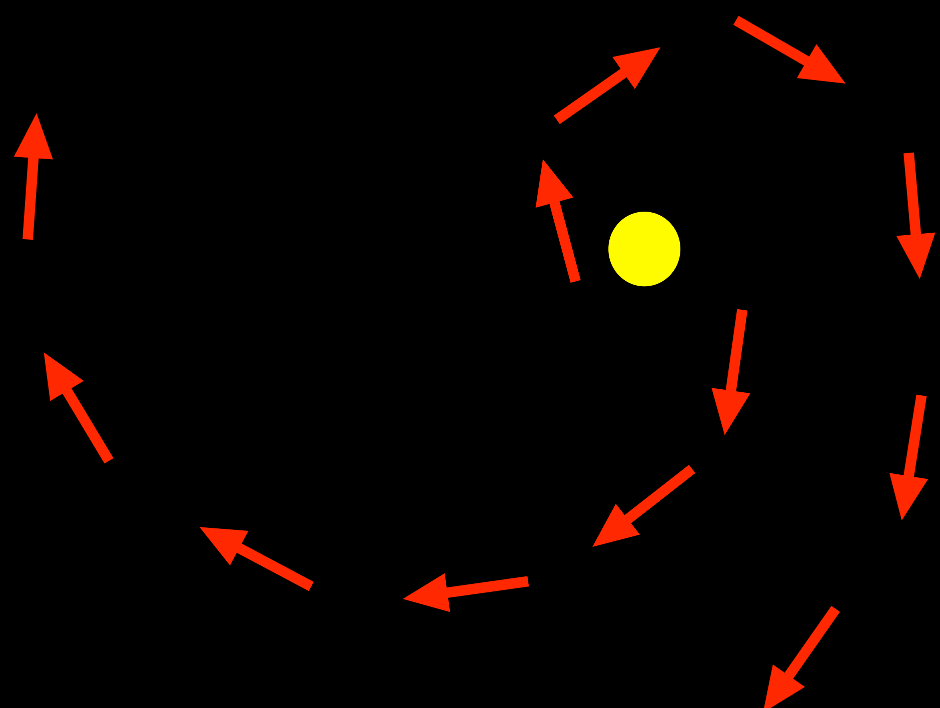
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$



$$a^2 + b^2 < 1$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$a^2 + b^2 > 1$$



Discrete dynamical system case

Asymptotic stability:

All eigenvalues satisfy

$$|\lambda_k| < 1$$

Material
since second midterm

We covered a lot since the second midterm!

But we like it extreme!

Systems of linear differential equations

WANTED! n functions

$$
\begin{aligned}
\frac{d}{dt}x_1(t) &= a_{11}x_1(t) + \ldots + a_{1n}x_n(t) \\
\frac{d}{dt}x_2(t) &= a_{21}x_1(t) + \ldots + a_{2n}x_n(t) \\
&\ldots \\
\frac{d}{dt}x_1(t) &= a_{n1}x_1(t) + \ldots + a_{nn}x_n(t)
\end{aligned}
$$

$$\dot{x} = Ax, \quad x(0)=v$$

We could write solutions as follows

$$x(t) = e^{At}\, v$$

but it does not give us much insight.

Here is half of what you have to know about differential equations

the mother of all differential equations:

$$\frac{d}{dt}\, x(t) = a\, x(t)$$

$$x(0) = b$$

It has the solution

$$x(t) = b\, e^{at}$$

and if you should not know:

And here is the second half of what you have to know about differential equations

x(t) = cos(kt) and x(t) = sin(kt)
satisfy
the differential equation

$$\ddot{x} = -k^2 x$$

The general solution is
A cos(k t) + B sin(k t)

it appears in mechanics: using Newtons law

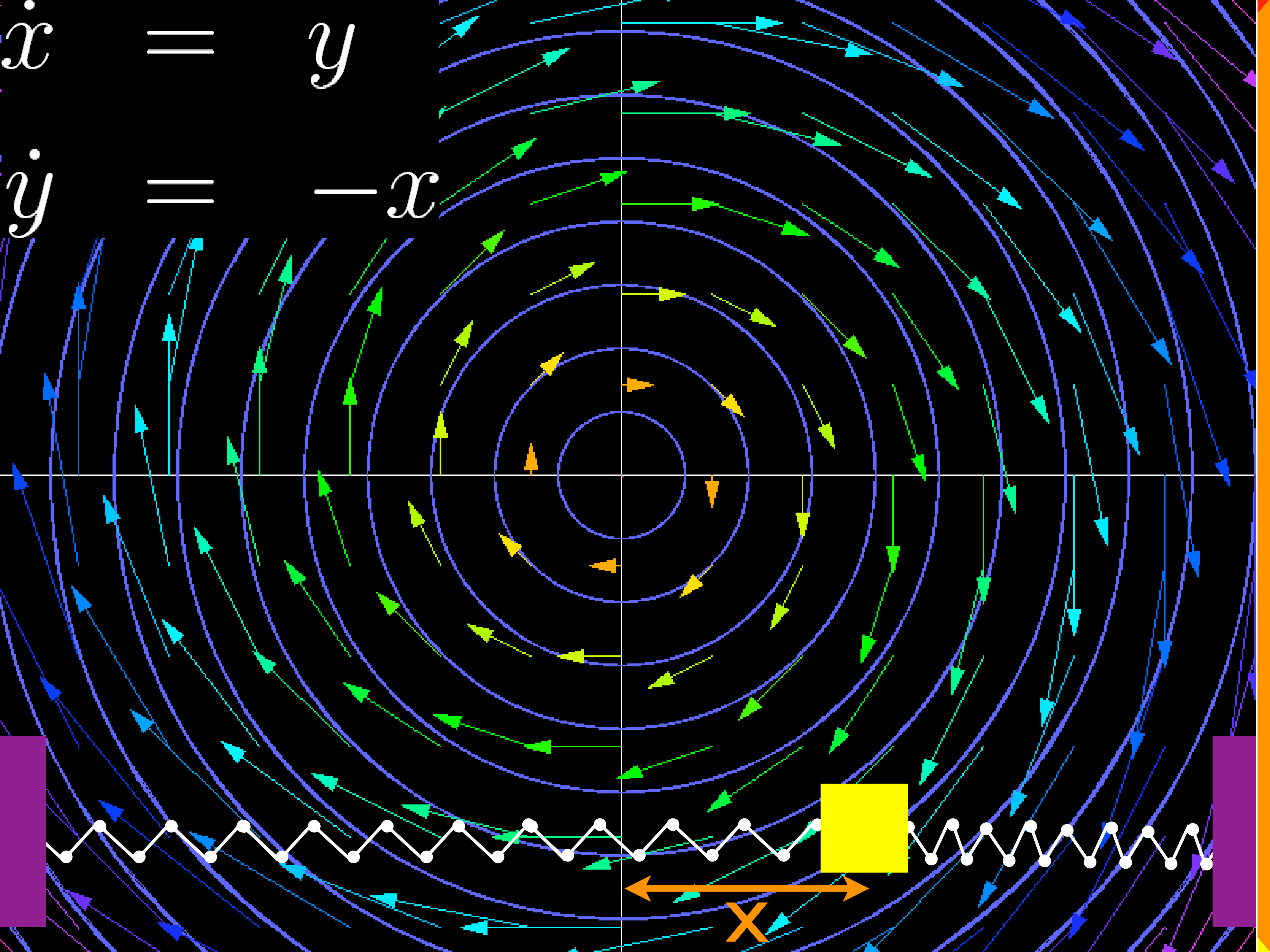
$$m\ddot{x} = -k^2 x$$

$$x(t) = A\cos(k\,t) + B\sin(k\,t)$$

This equation is called the harmonic oscillator It is extremely important in physics.

$$\dot{x} = y$$

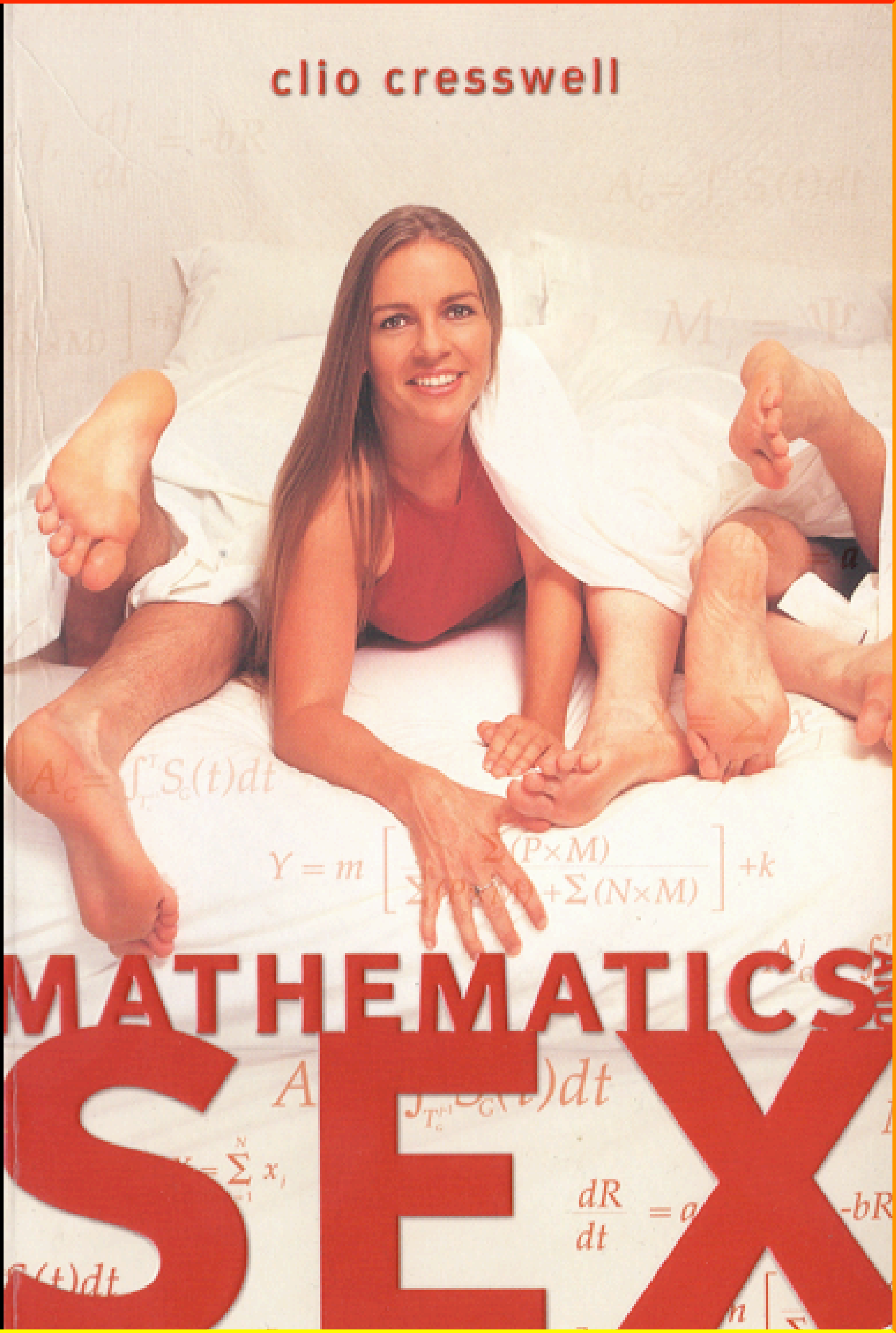
$$\dot{y} = -x$$



but it is also relevant in love relationships and since our review already has this theme, why not stick to it:

The harmonic oscillator appears also in the book of

Clio Cresswell
2003



Not in the most obvious way although...

Here is part of the book

LOVE, SWEEEET LOVE

In the late '80s, a Harvard lecturer by the name of Steven Strogatz suggested an unusual class exercise to his students. The day's topic would be the Mathematics of Love. Professor Strogatz's motivations were plain cheeky. Confronted with the challenge of capturing his students' attention on the predictive powers of equations, he reworded a common undergraduate mathematics problem into a language he thought the students would relate to: the evolution of the love affair between Romeo and Juliet. His ingenuity should not be taken lightly: turning a group of hormone-raging twentymake some mathematical sense of one of the great human emotions.

He presented the problem like this:

> Romeo is in love with Juliet, but in our version of the story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

As you can see, emotions are a bit all over the place in this relationship. The question is, will they ever settle? What kind of relationship can Romeo and Juliet look forward to? The point

The first step towards mathematical insight is to rewrite the terms of Romeo and Juliet's fickle affair mathematically. The translation is:

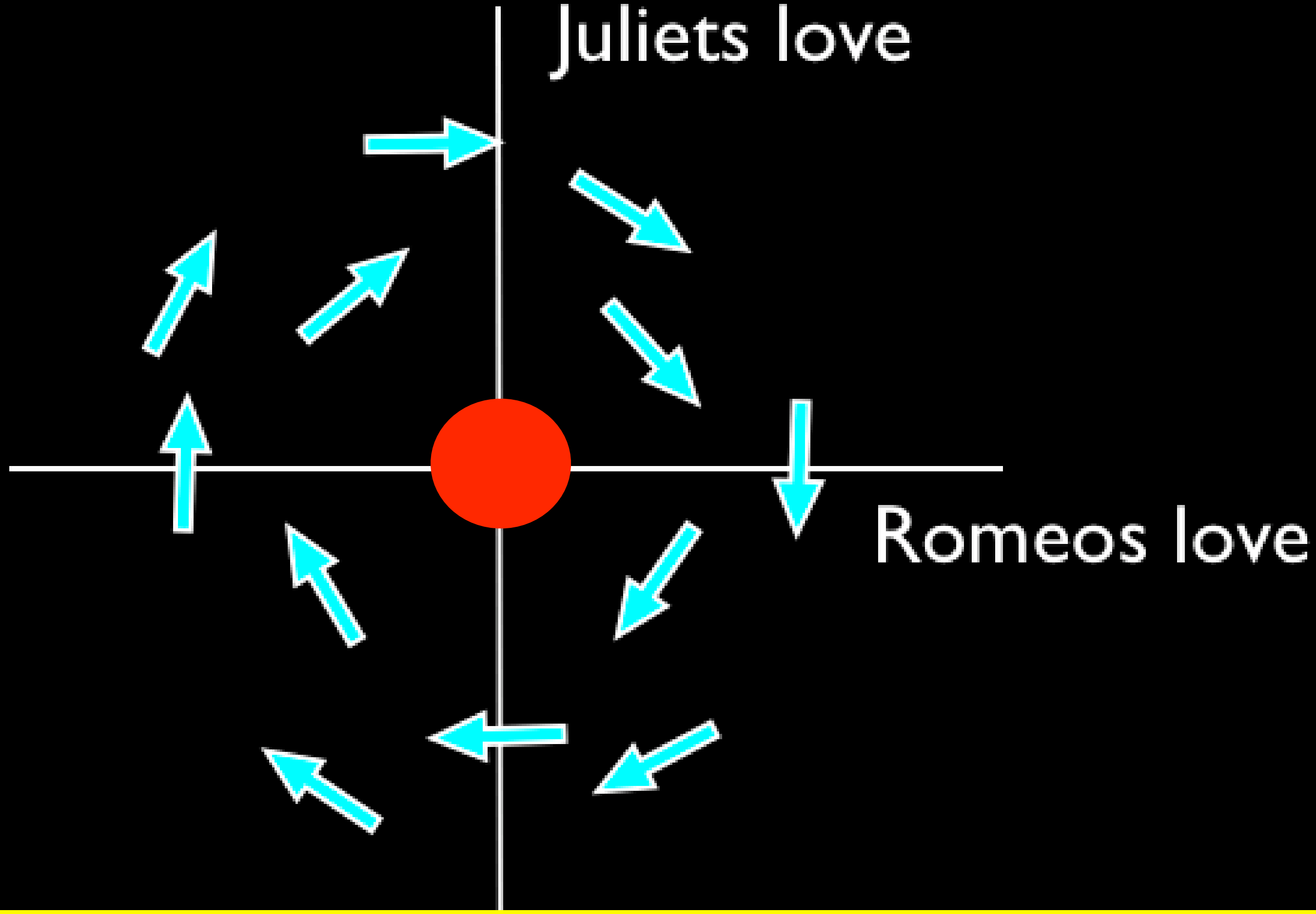
$$\frac{dR}{dt} = aJ, \frac{dJ}{dt} = -bR,$$

where R is for Romeo, and J for Juliet. How the letters are combined mimics how Romeo and Juliet find themselves interacting. For mathematicians, translating the problem into equations like this is natural. Mathematics is the study of patterns and this problem simply concerns behavioural patterns. Behavioural patterns are not static though and that's an important characteristic to

Romeo and Juliet

Romeo warms up when given more love, Juliet wants to run away when being desired.

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$



if we add a damping factor:

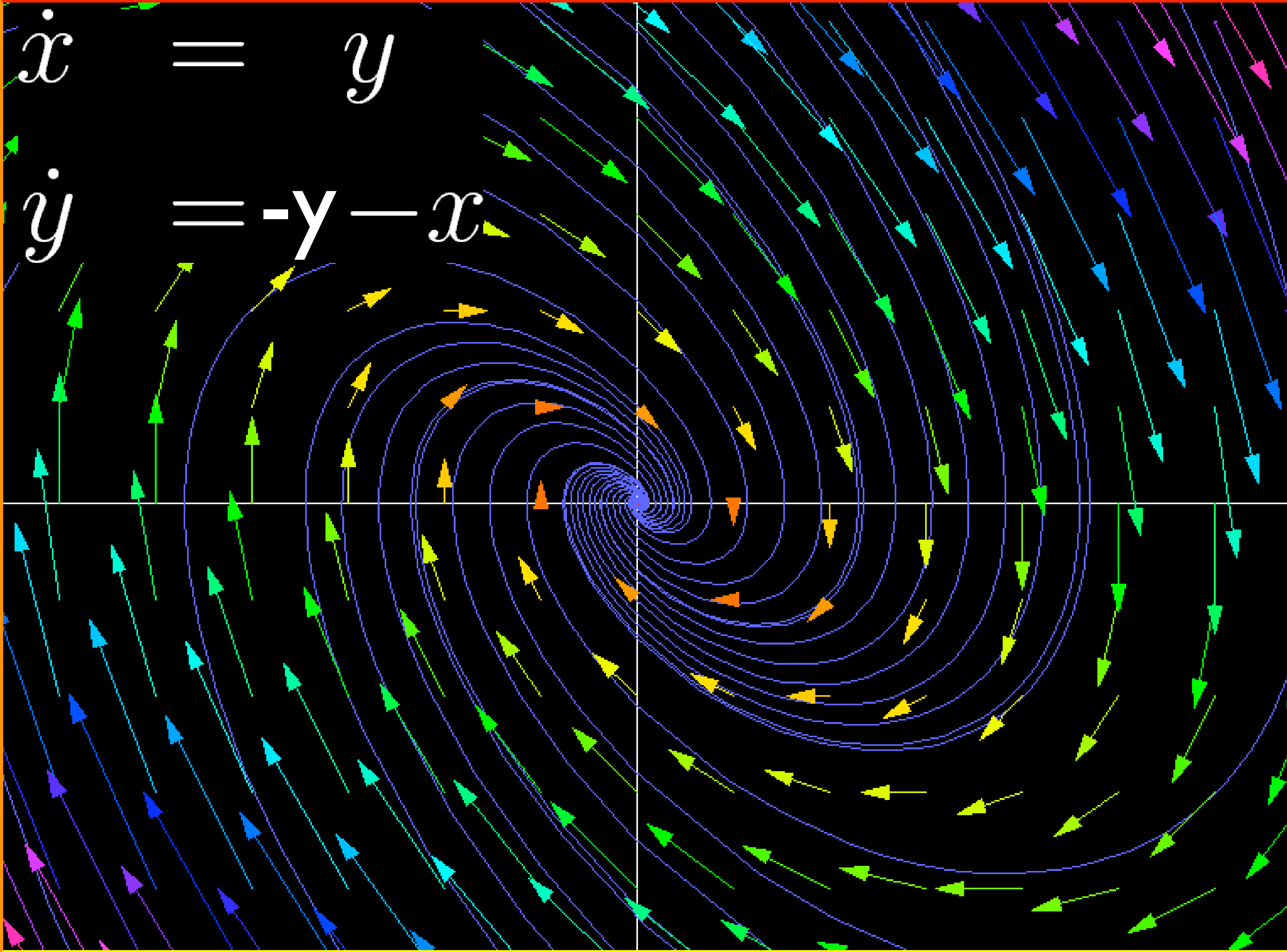
$$\ddot{x} = -b\dot{x}-k^2x$$

we have a situation relevant in love also as tabloids show us. Lets look at this later.

it is good to know that it has the solution

$$x(t) = e^{-bt} \left(A \cos(c\, t) + B \sin(c\, t) \right)$$

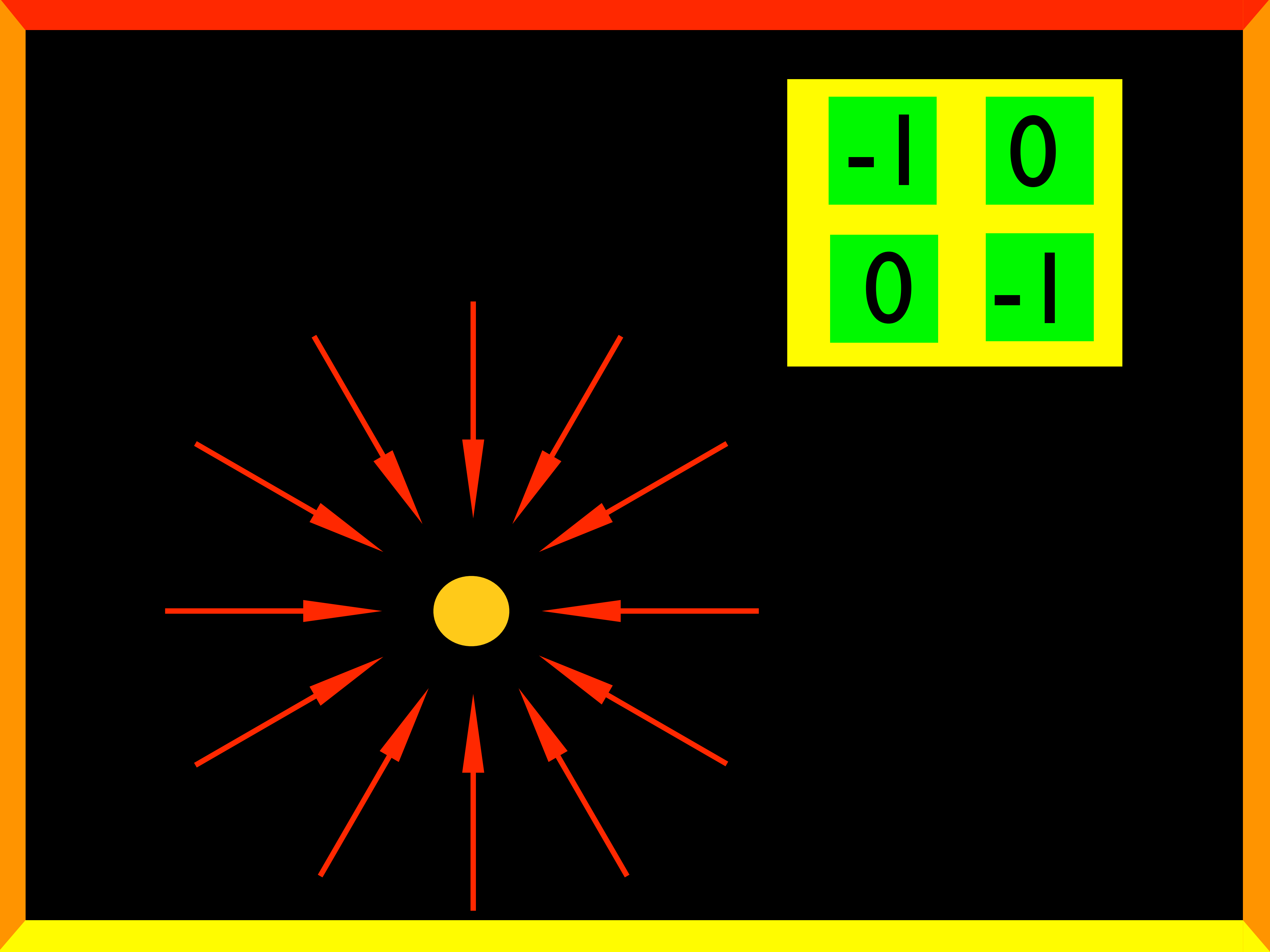
where $-b + i\, c$ are the roots of the system.


$\dot{x} = y$
$\dot{y} = -y - x$

Stability of continuous dynamical systems

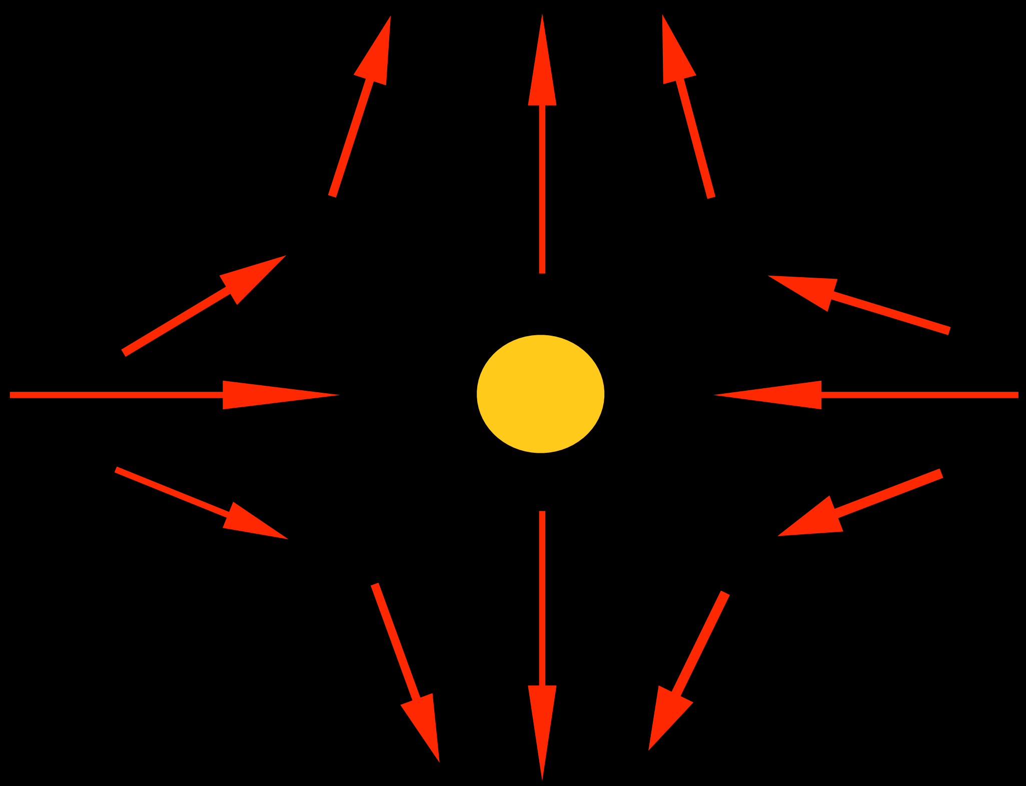
Now, the real part of the eigenvalues is relevant!

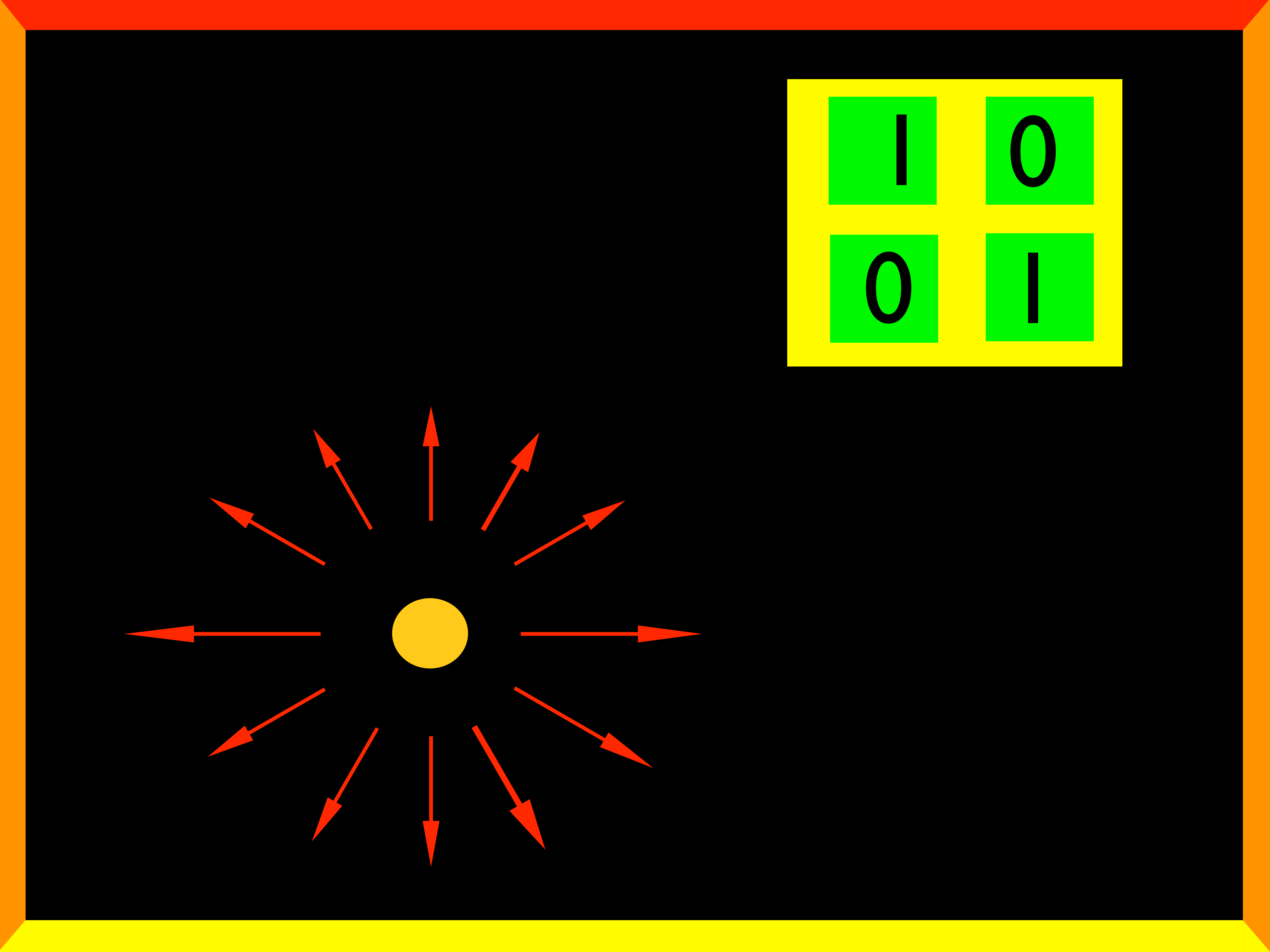
Proof: ask Mother!



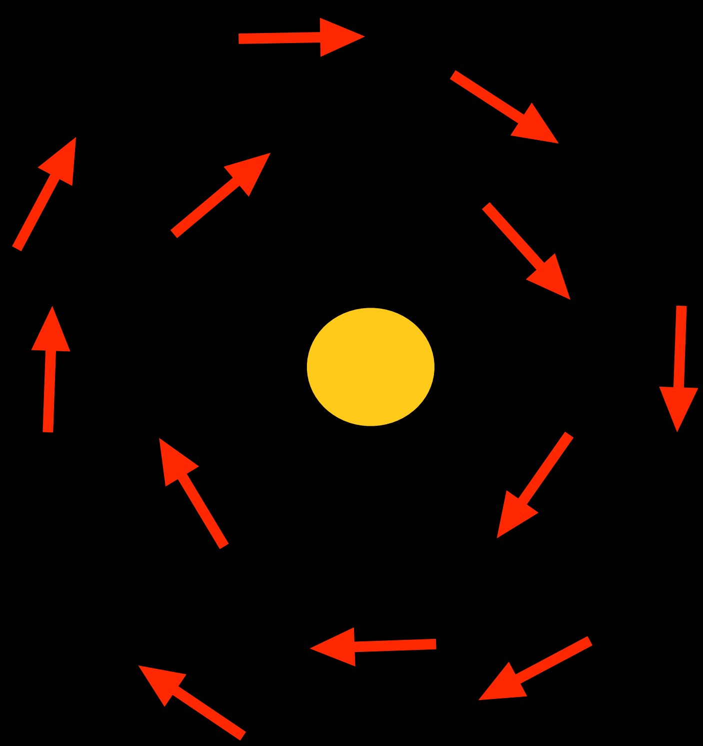
-1
0
0
-1

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

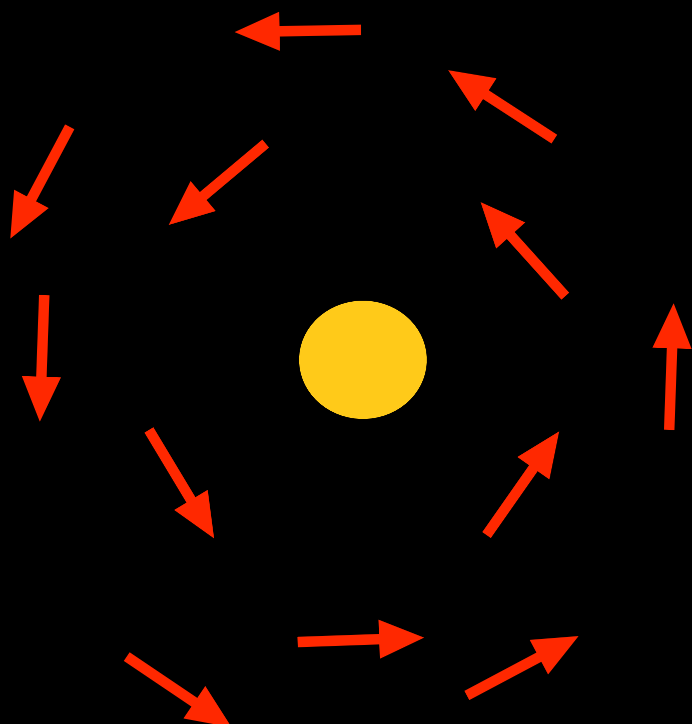



1 0
0 1

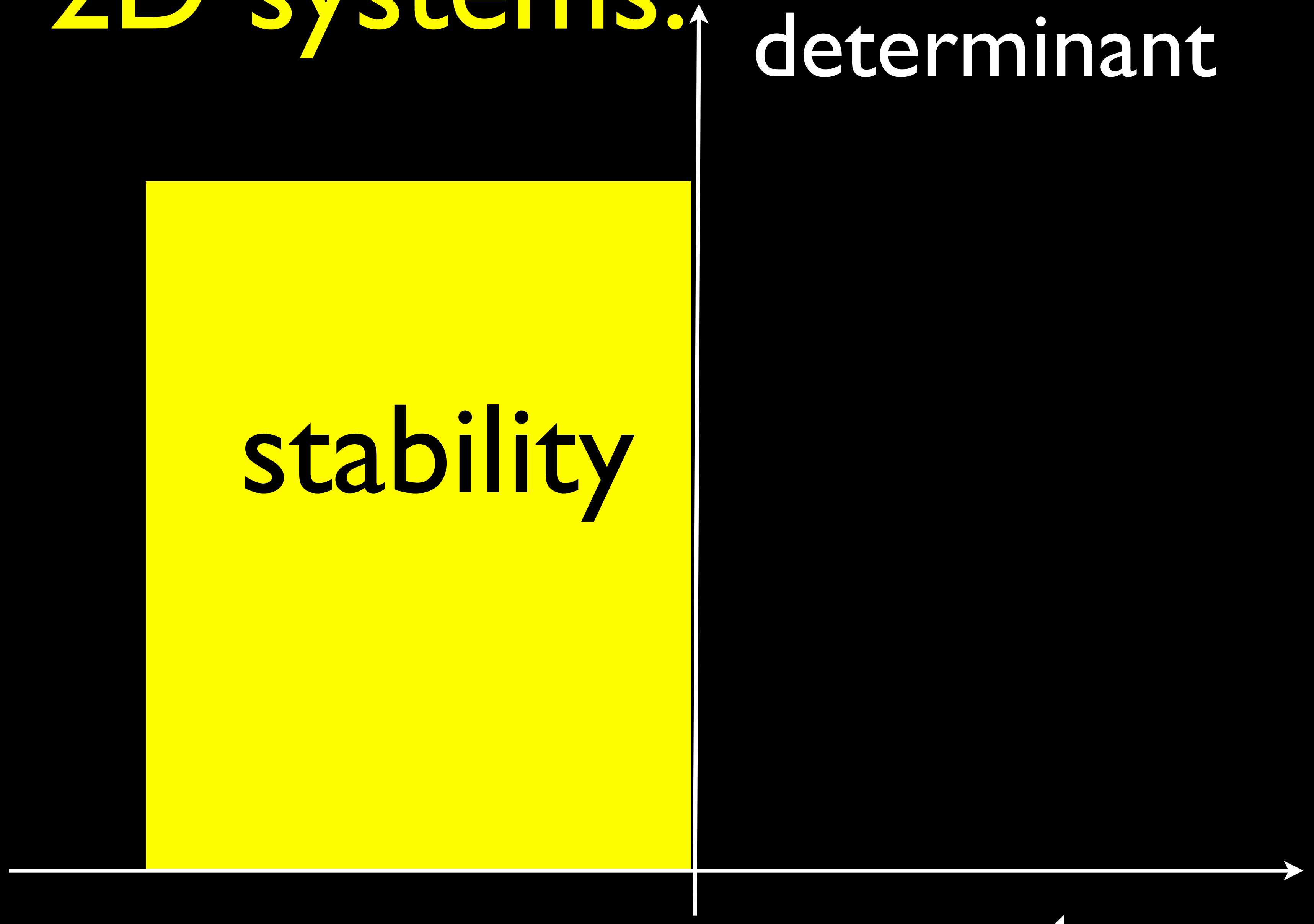
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



for 2D systems:



no chalk problem

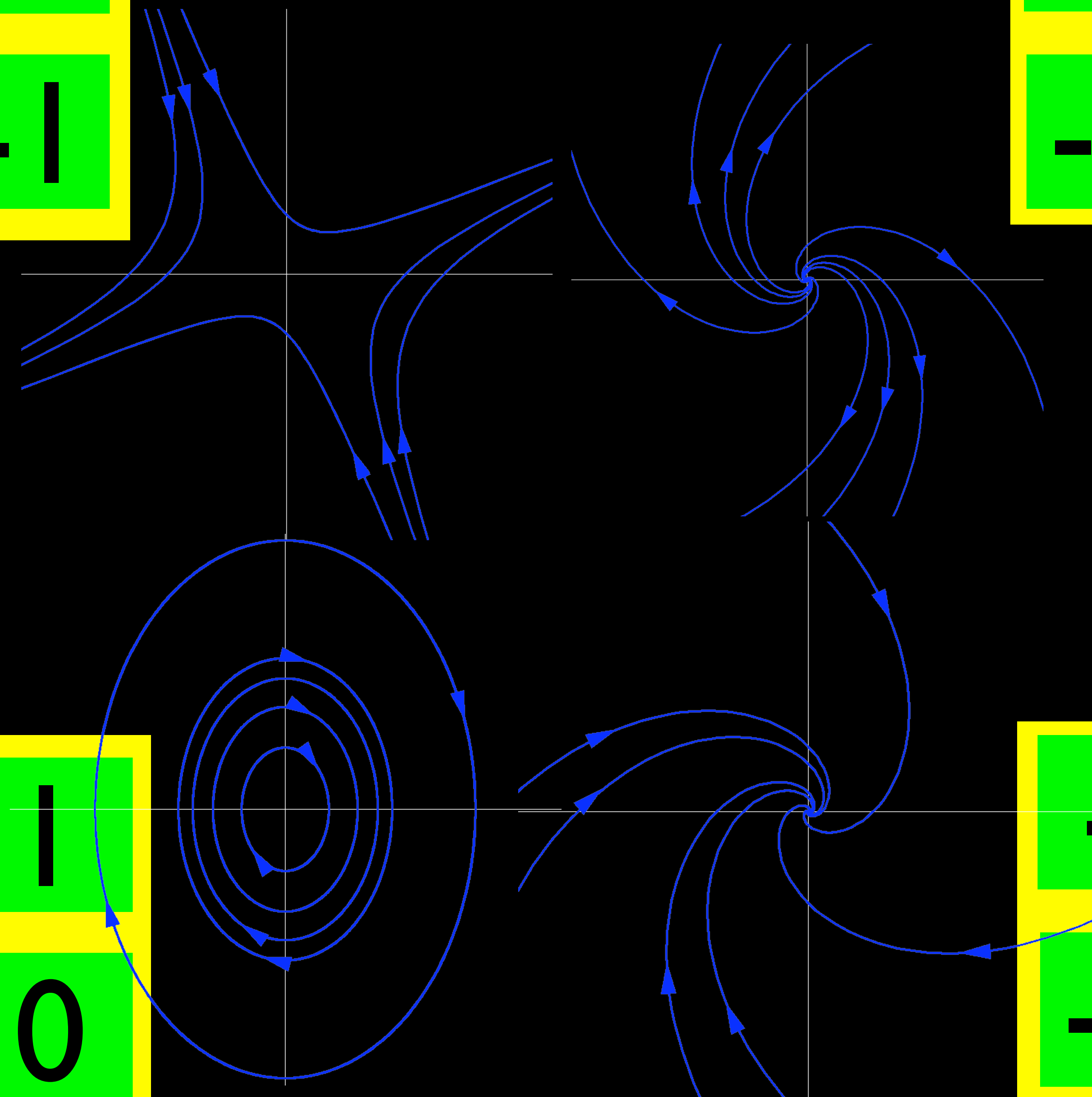
Determine the stability of the systems with the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

A forbidden matrix

AACS is a DRM scheme used to encrypt data on HD-DVD and Blu-Ray disks. There is only a small problem:

the key is known:

A forbidden matrix

Here it is:

09	f9	11	02
9d	74	e3	5b
d8	41	56	c5
63	56	88	c0

HEX

9	249	17	2
157	116	227	91
216	65	86	197
99	86	136	192

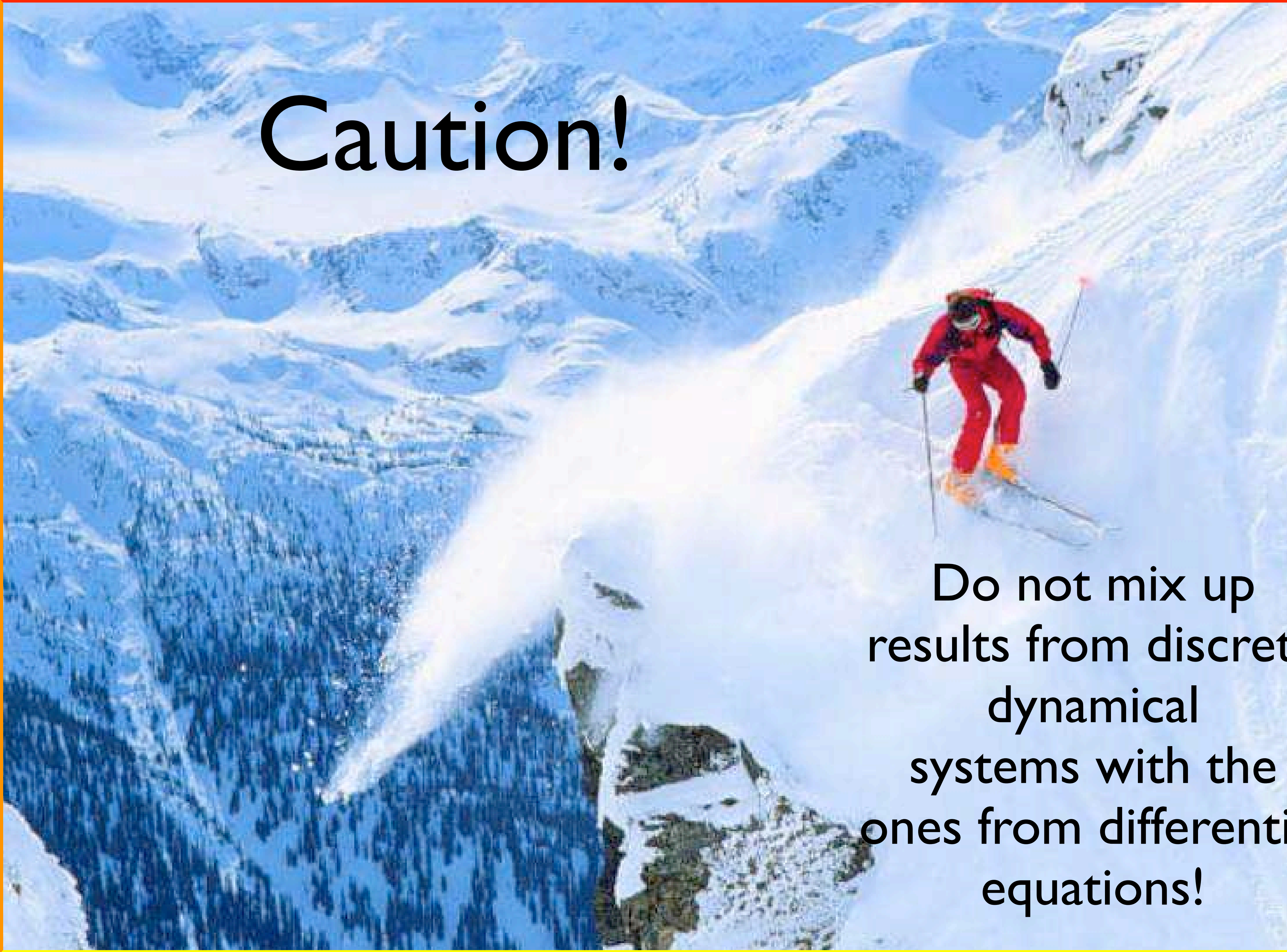
DECIMAL

no chalk problem

The key leads to a forbidden system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 & 249 & 17 & 2 \\ 157 & 116 & 227 & 91 \\ 216 & 65 & 86 & 197 \\ 99 & 86 & 136 & 192 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Is the system stable?


Caution!
Do not mix up results from discret dynamical systems with the ones from differenti equations!

Inhomogeneous differential equations

this is one of the most challenging topics. There are two different ways to solve it:

operator method

cookbook method

Differential operators

$$D f = f'$$

$p(D)$ polynomial

$$p(D)=(D-\lambda_1)\ldots(D-\lambda_n)$$

fundamental theorem of algebra

To solve a inhomogeneous differential equation

$$p(D)\, f = g$$

factor p(D) into linear factors and invert each linear factor

$$f=(D-\lambda_1)^{-1}\ldots(D-\lambda_n)^{-1}g$$

using the inversion formula for linear differential operators

$$(D-\lambda)^{-1}g = ce^{\lambda x} + e^{\lambda x}\int_0^x e^{-\lambda t} g(t)\, dt$$

Allows to solve all problems without exception, but needs integration skills.

there is also a cookbook method

Cookbook

Solution of $p(D)\, f = g$

Solve the homogeneous problem $p(D)\, f = 0$

Find a special solution using an educated guess

For second order equations, there are three cases

- different real roots
- same real roots
- complex conjugated roots

blackboard problems

Oscillations

$$f'' + 9 f = 2 \sin(2 t)$$

Homogeneous:	Inhomogeneous:
$C_1 \cos(3t) + C_2 \sin(3t)$	$A \cos(2t) + B \sin(2t)$

$9 A \sin(2t) + \quad 5 B \sin(2 t) = 2 \sin(2 t)$ gives A,B

Example:

$$\ddot{f} - 3\dot{f} + 2f = 1$$

Mother tells us: try

$$f(t) = e^{at}$$

$$(a^2 - 3a + 2)e^{at} = 0$$

$a=2$ or $a=1$ and e^t and e^{2t} are solutions.

for the special solution: try $f = a$

Example with operator method

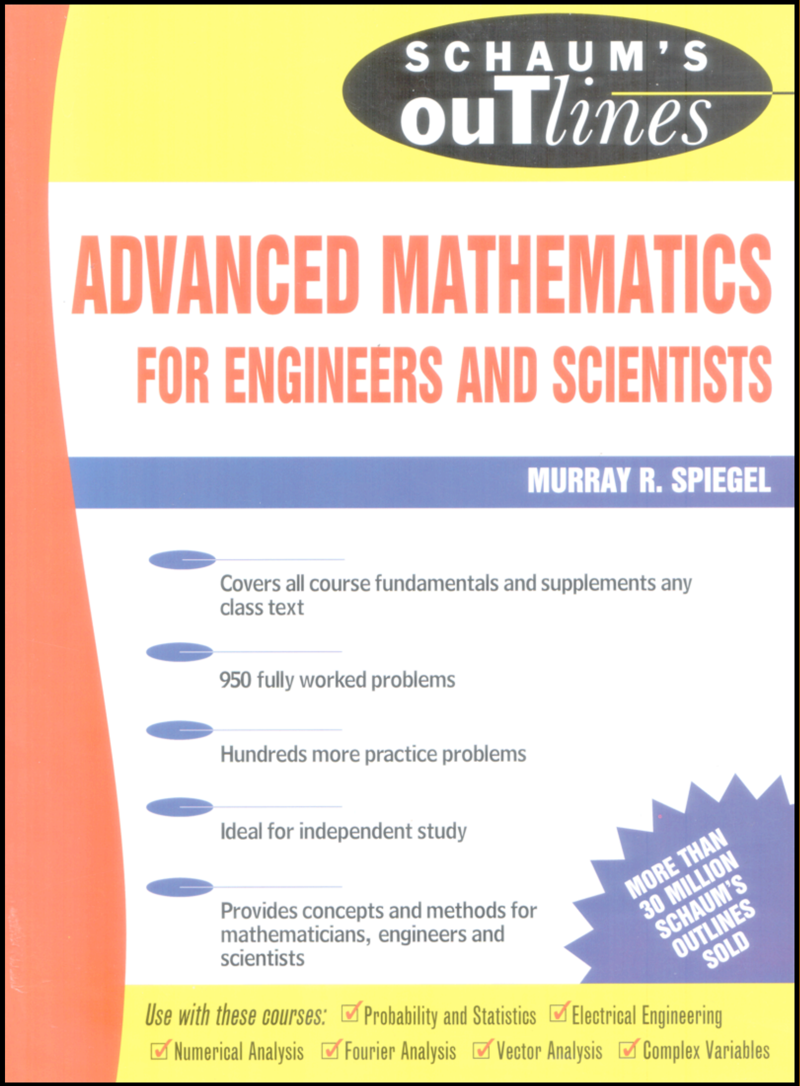
$$f'' + 10 f' + 21 f = 1$$

$$(D+3)(D+7) f = 1$$

$$(D+3)^{-1} t = A e^{-3t} + 1/3$$

$$(D+7)^{-1} [A e^{-3t} + 1/3] = A e^{-3t} + B e^{-3t} + 1/21$$

The cookbook method is popular among engineers and is often faster than the operator method



How do we guess the special solution?

right hand side	Try with
sin(kt)	A sin(kt) + B cos(kt)
$e^{k t}$	$e^{k t}$
1	c
t	at+b
t^2	$at^2+bt + c$

if already homogenous solution, take $te^{k t}$ At sin(kt) + B t cos(kt)

more blackboard problems

4 examples

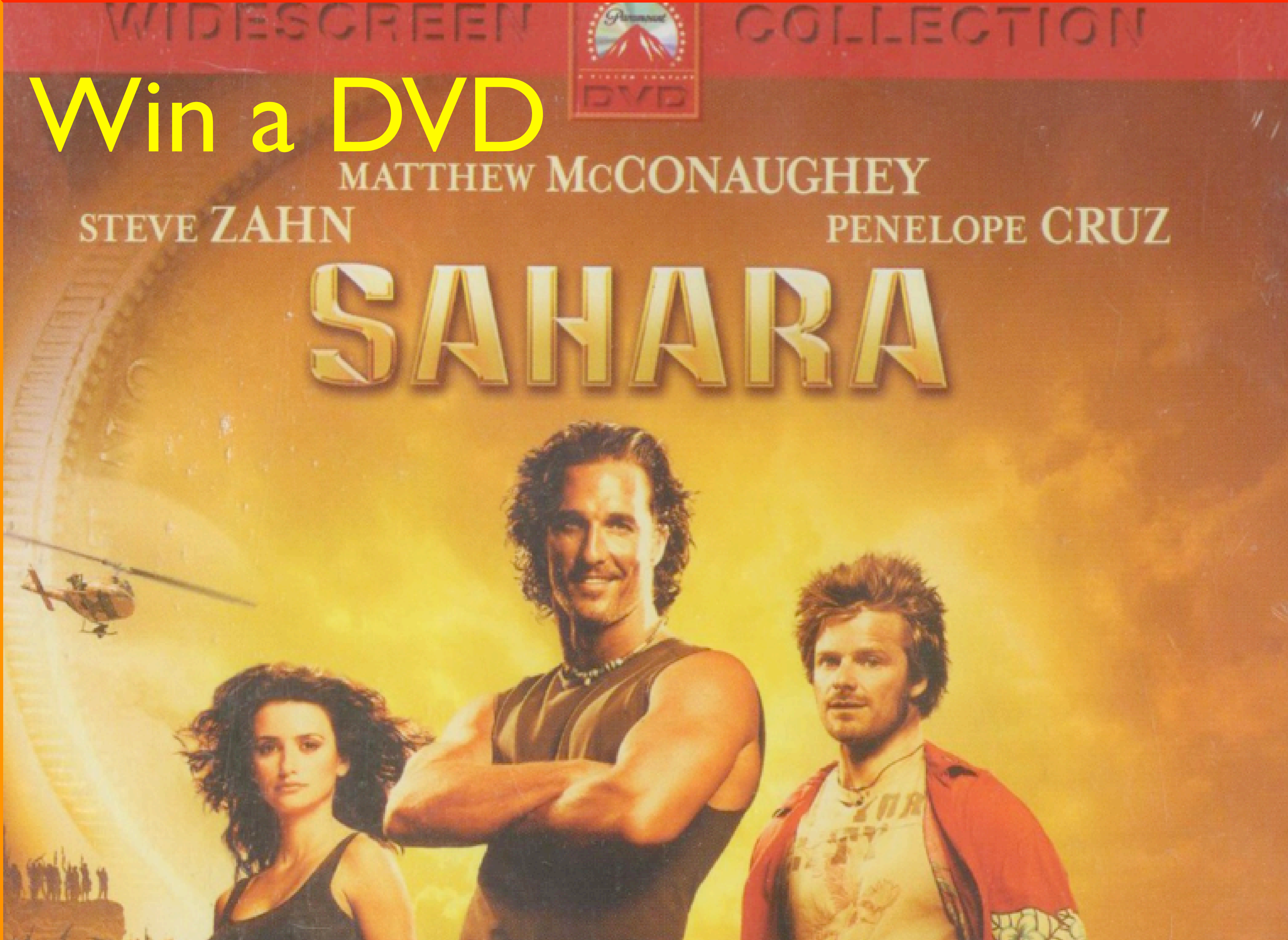
$$f' - 3f = e^t$$

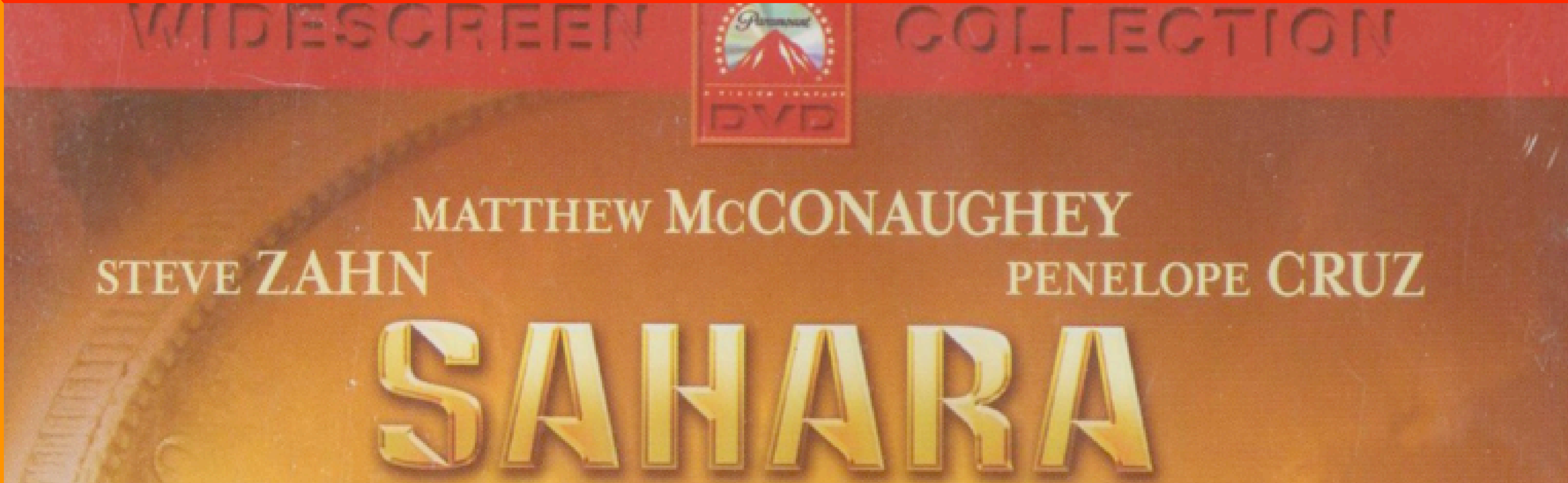
$$f'' - 6f' + 9f = e^t$$

$$f'' + 9f = e^t$$

$$f'' + 2f' + f = t$$

Win a DVD

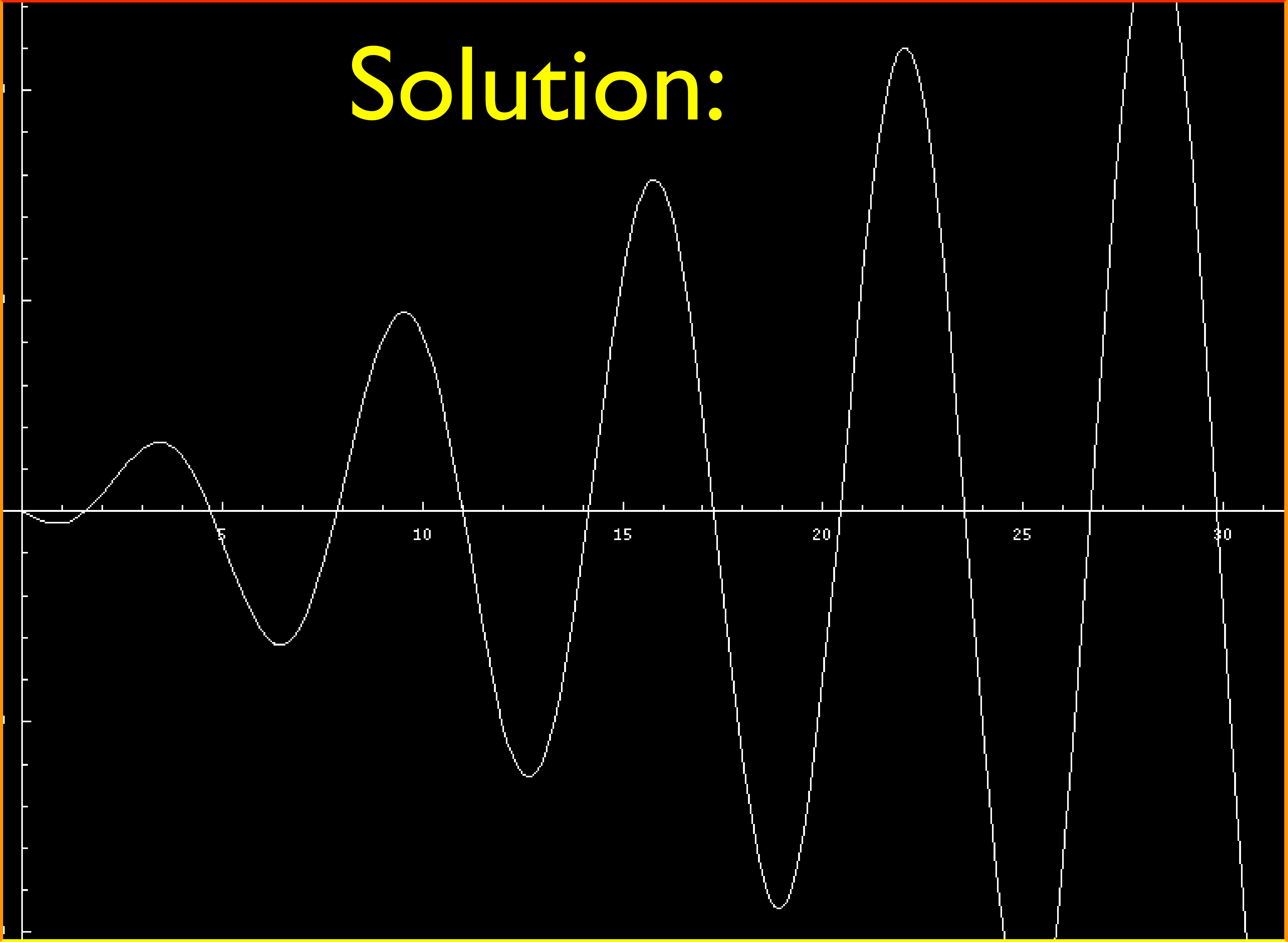

WIDESCREEN COLLECTION
Paramount
DVD
MATTHEW McCONAUGHEY
STEVE ZAHN
PENELOPE CRUZ
SAHARA


WIDESCREEN
COLLECTION
Paramount
DVD
MATTHEW McCONAUGHEY
STEVE ZAHN
PENELOPE CRUZ
SAHARA

Find a special solution of

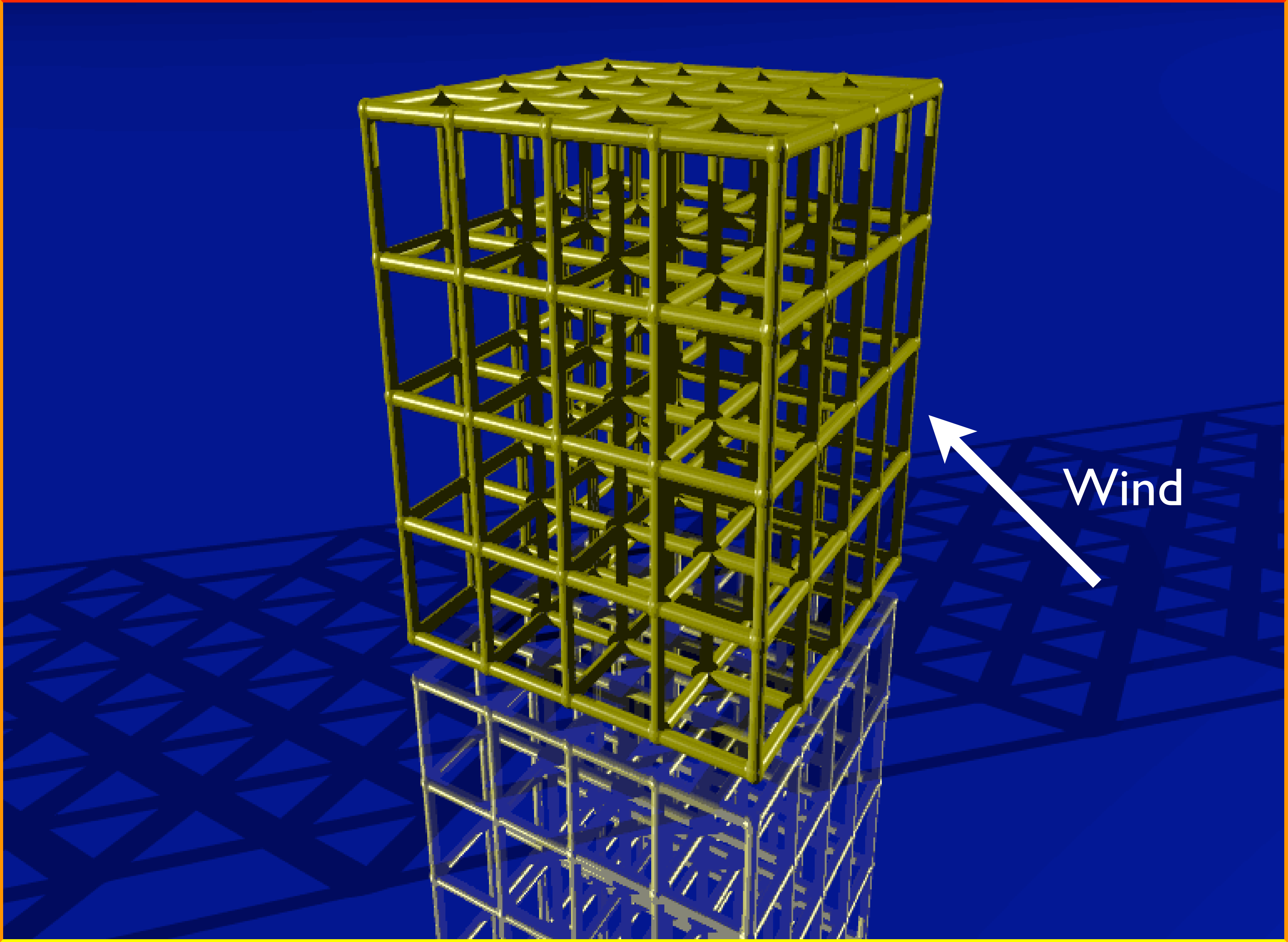
$$f''(t) + f(t) = 2 \sin(t)$$

Solution:



Apropos resonance: the Hancock Tower

- 790 feet=241 m
- 60 story
- 300 ton mass dampeners
- constructed 1973-1976
- 45th tallest US building
- 131 tallest world
- 16 Herz motion


Wind

Nonlinear systems

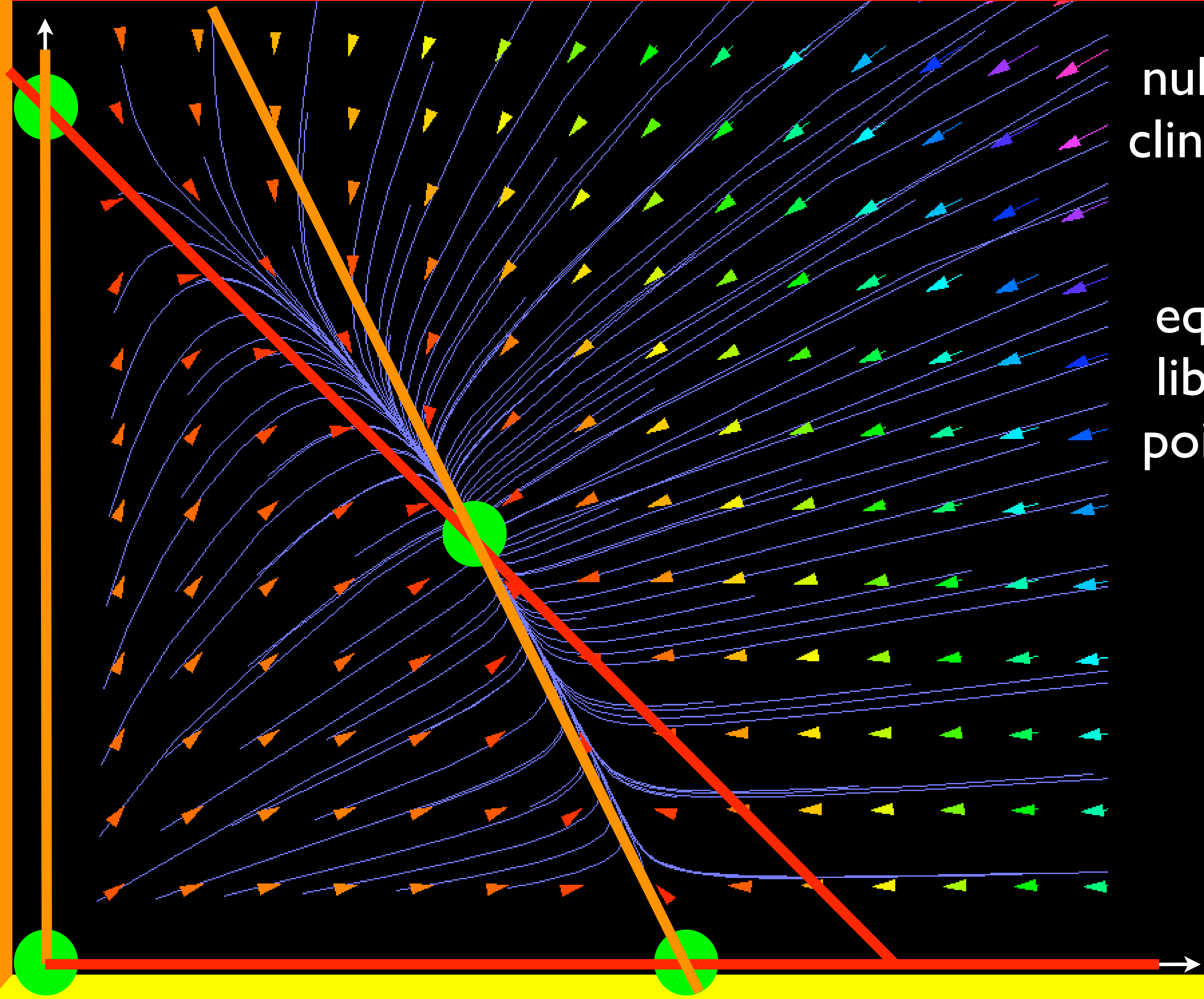
We look at equations in the plane

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

The example from the handout:

$$\dot{x} = x(6-2x-y)$$

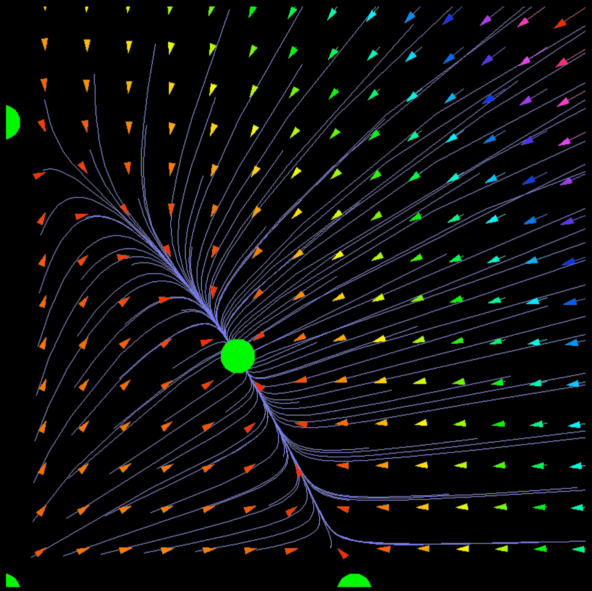
$$\dot{y} = y(4-x-y)$$



null-
clines

equi-
libria
points

Jacobean matrix



$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

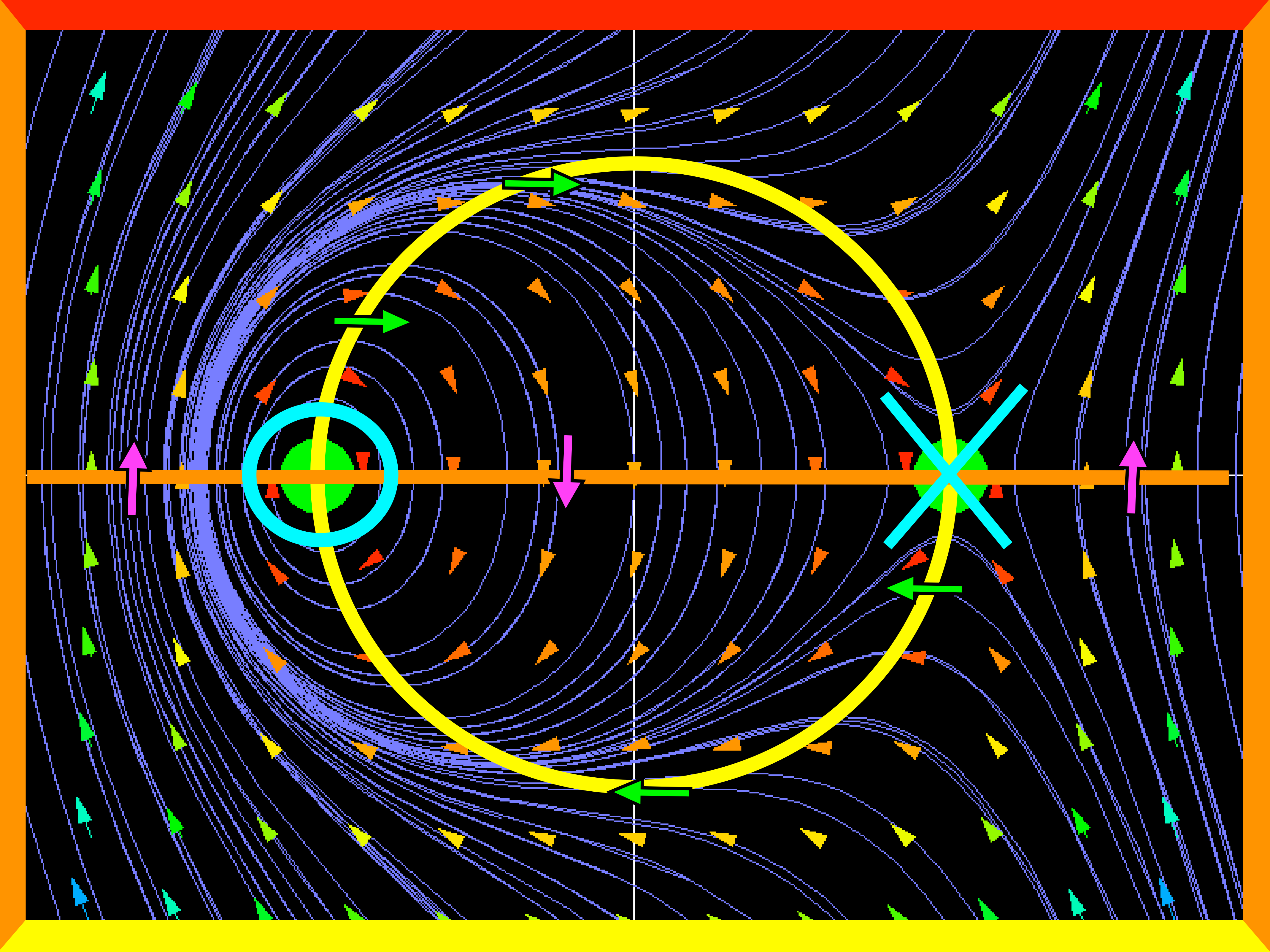
$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

blackboard problems

Analyze

$$\dot{x} = y$$

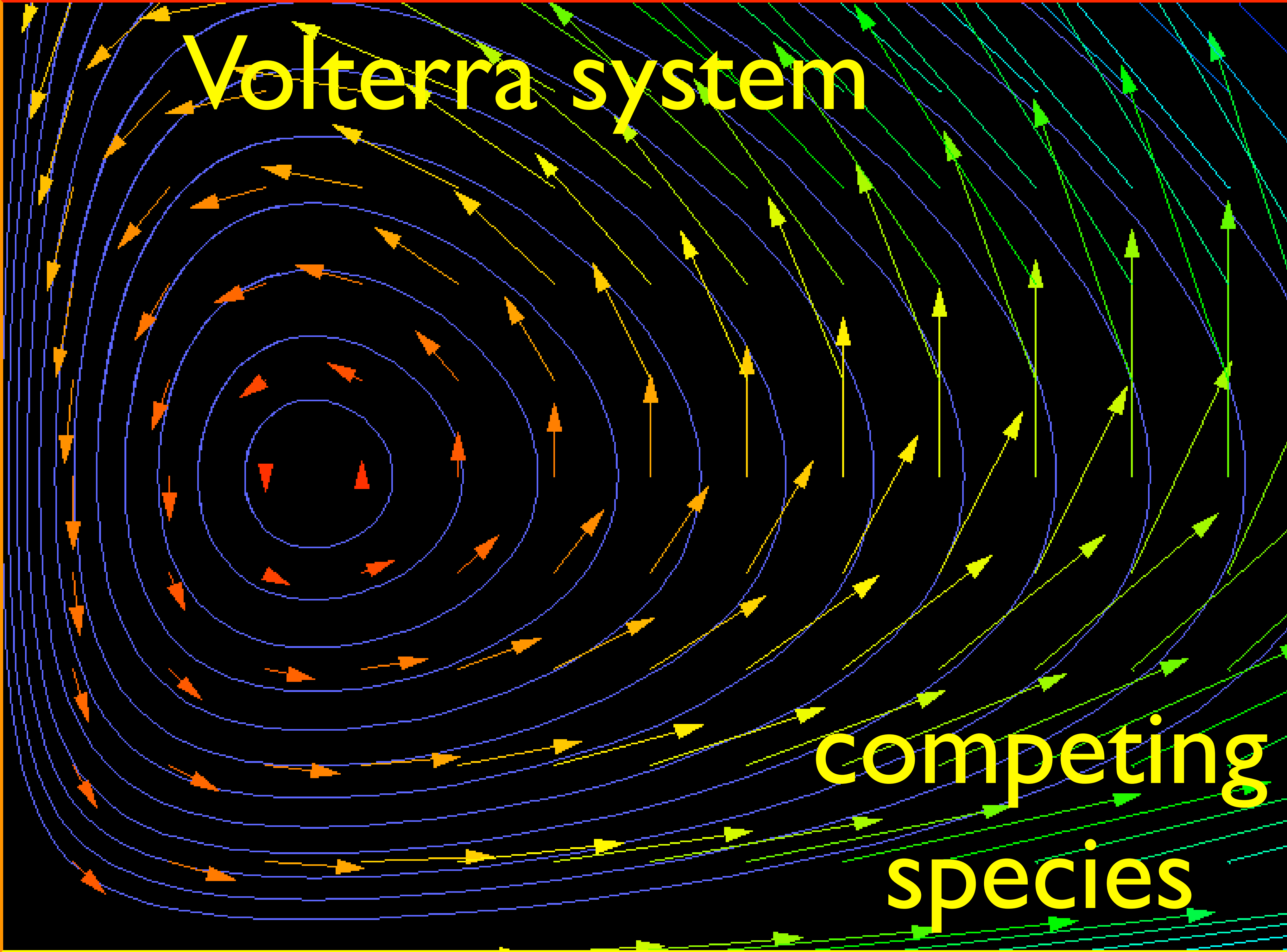
$$\dot{y} = x^2 + y^2 - 1$$



More examples (only as illustration)

Volterra system

$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$

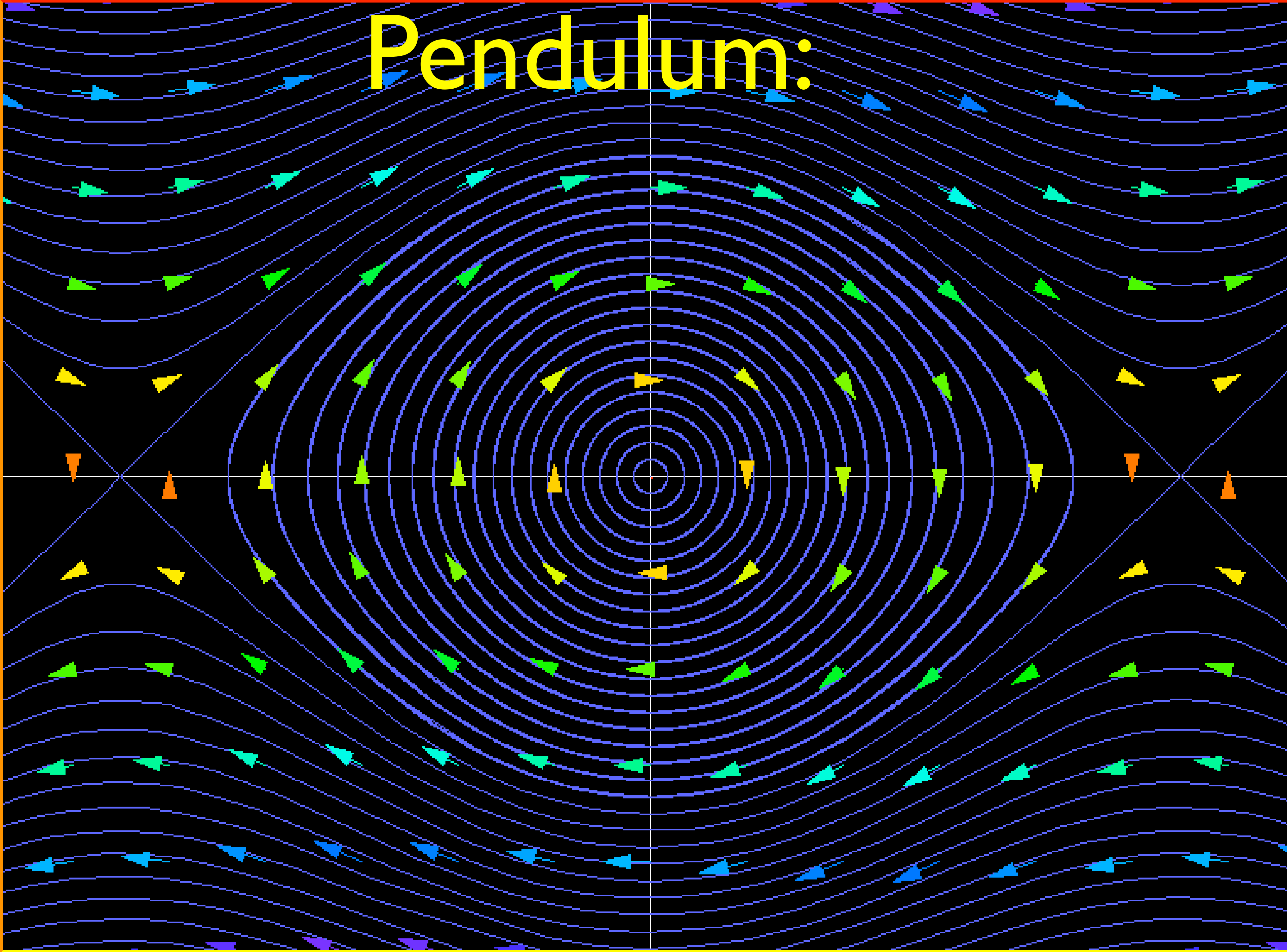

Volterra system
competing
species

Pendulum from homework

$$\dot{x} = y$$

$$\dot{y} = -\sin(x)$$

Pendulum:

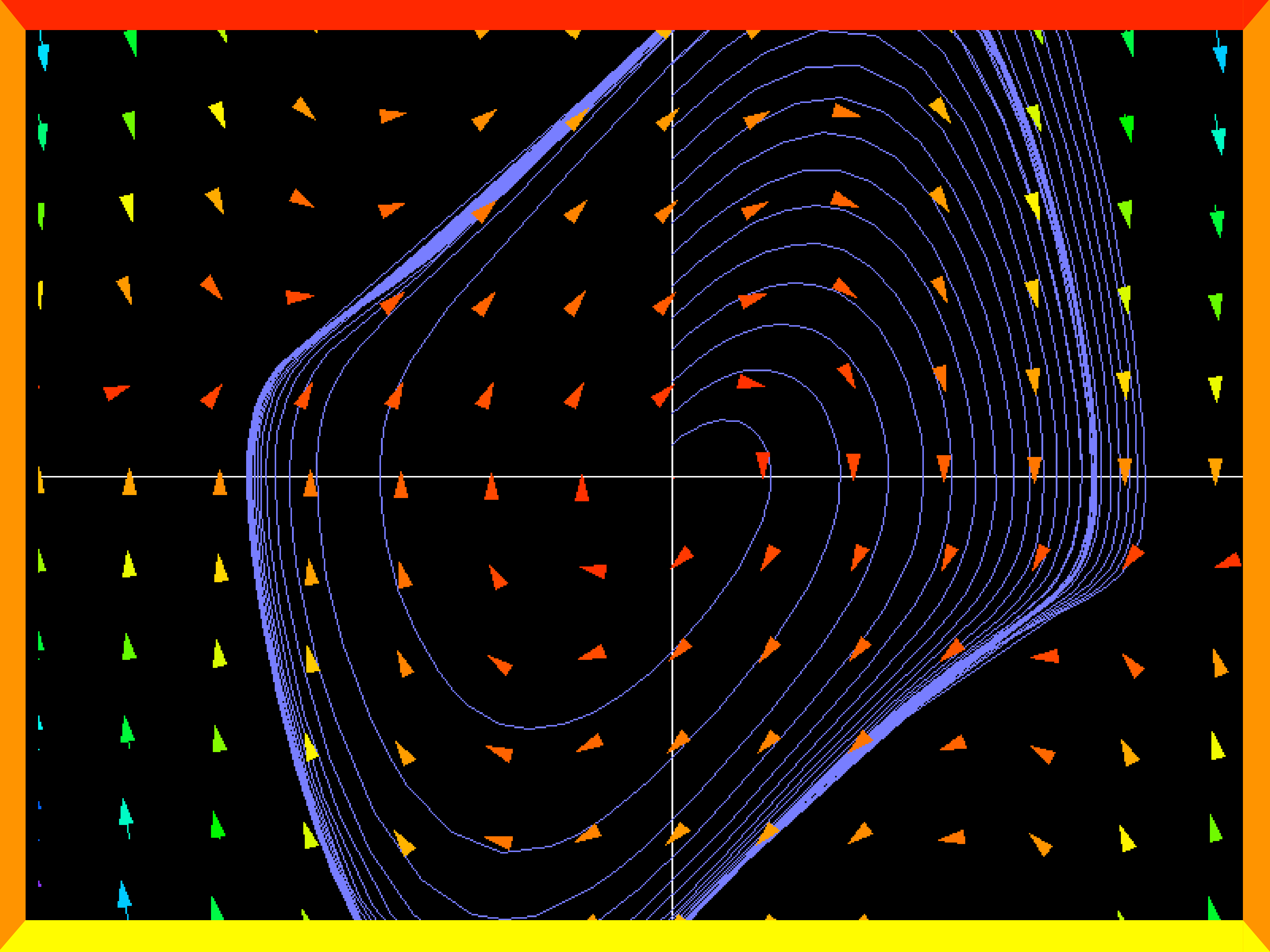


Van der Pool system

$$\dot{x} = y$$

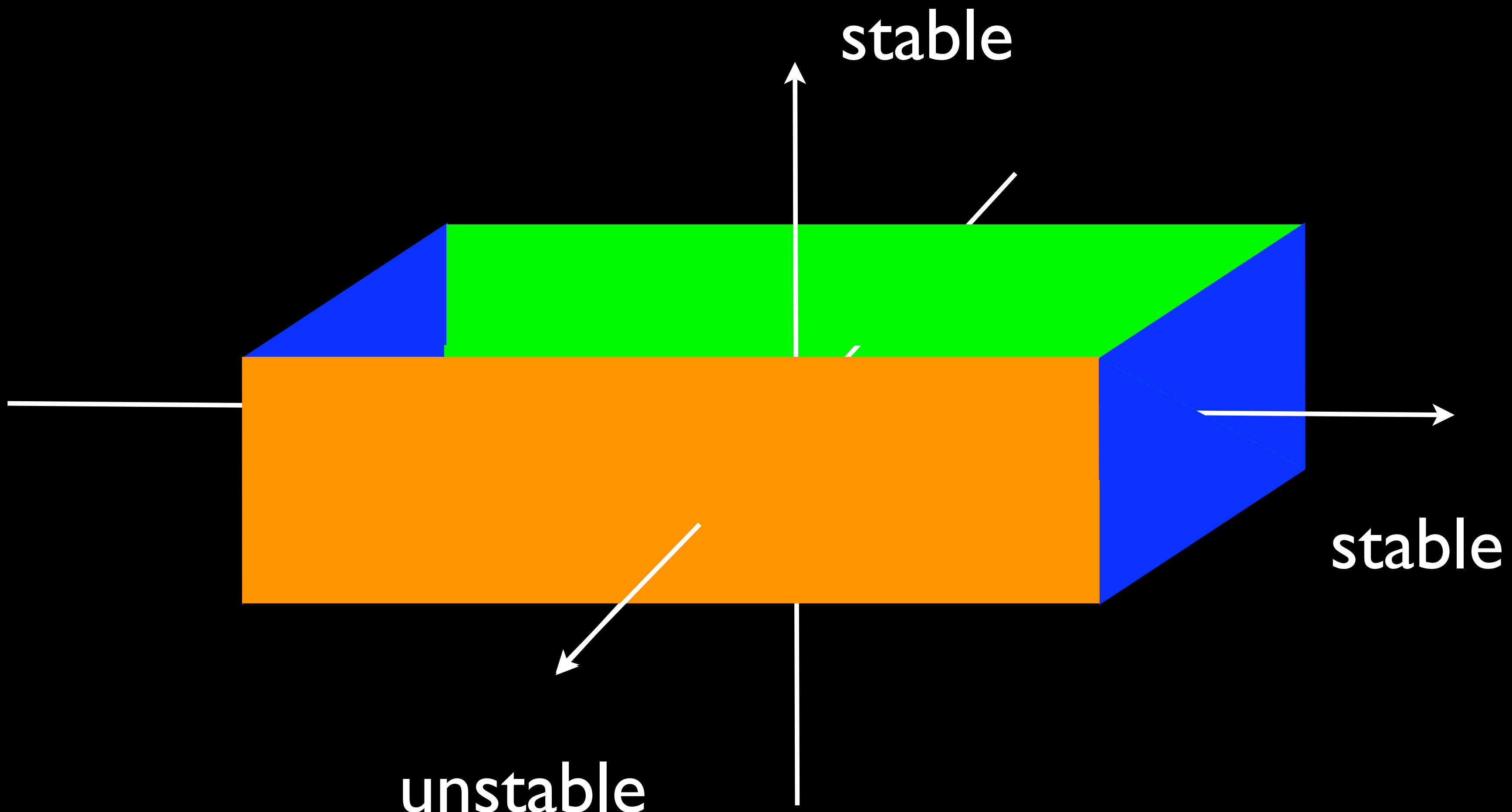
$$\dot{y} = -x - (x^2 - 1)y$$

nonlinear oscillator where damping depends on x



Here is a problem in
three dimensions
which is mentioned in
this review
only to show the
power of linarization

Spinning book



Differential equations

$$A\dot{x} = (B-C)\, y\, z$$

$$B\dot{y} = (C-A)\, z\, x$$

$$C\dot{z} = (A-B)\, x\, y$$

x,y,z angular velocities, A,B,C moment of inertias

Linearize around x=1,y,z=0

$$A\dot{x} = 0$$

$$B\dot{y} = (C-A)\,z$$

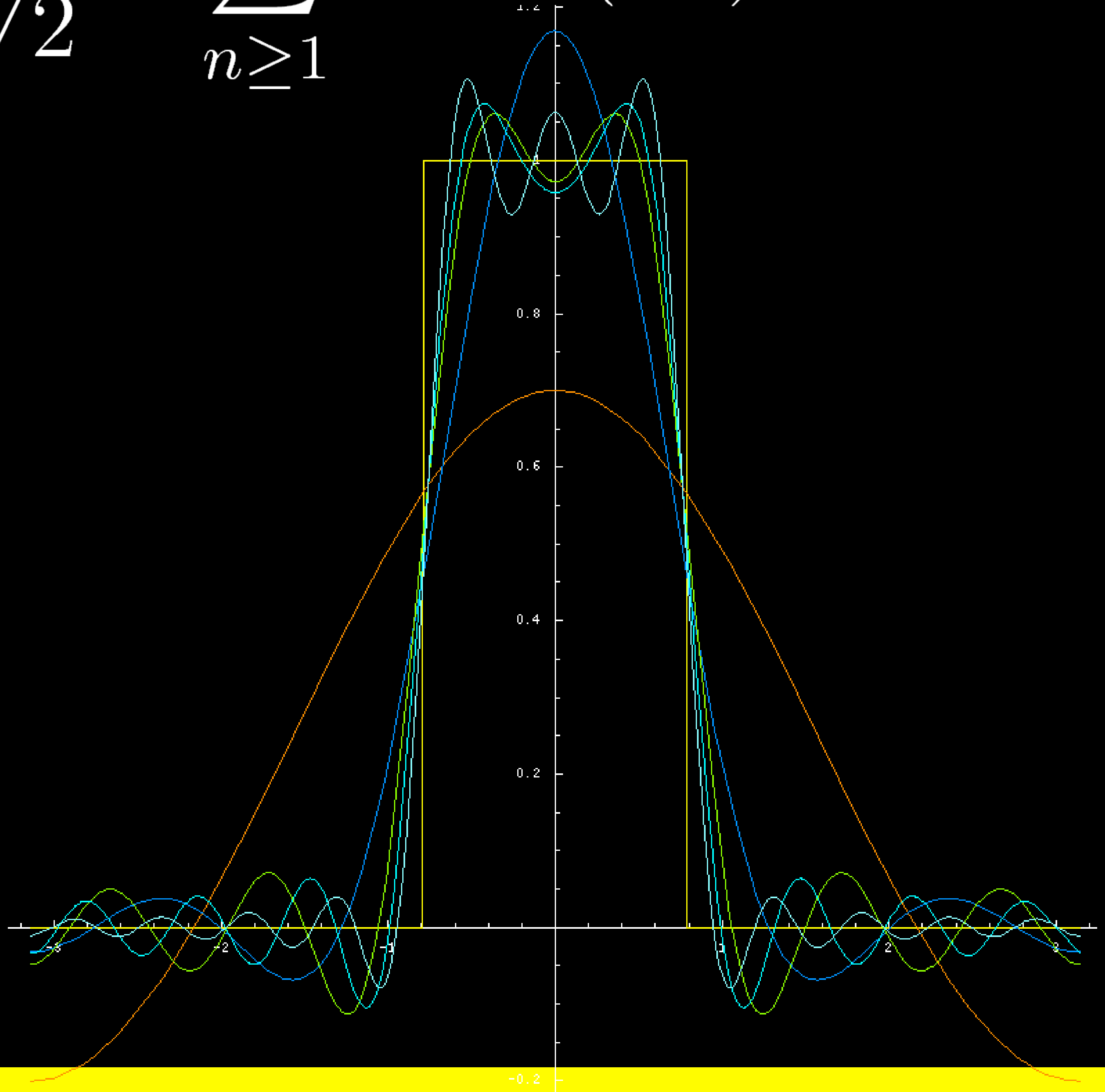
$$C\dot{z} = (A-B)\,y$$

$$BC\ddot{y} = (C-A)\,(A-B)\,y$$

unstable if A is between B and C. Otherwise stable.

Fourier analysis

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$$



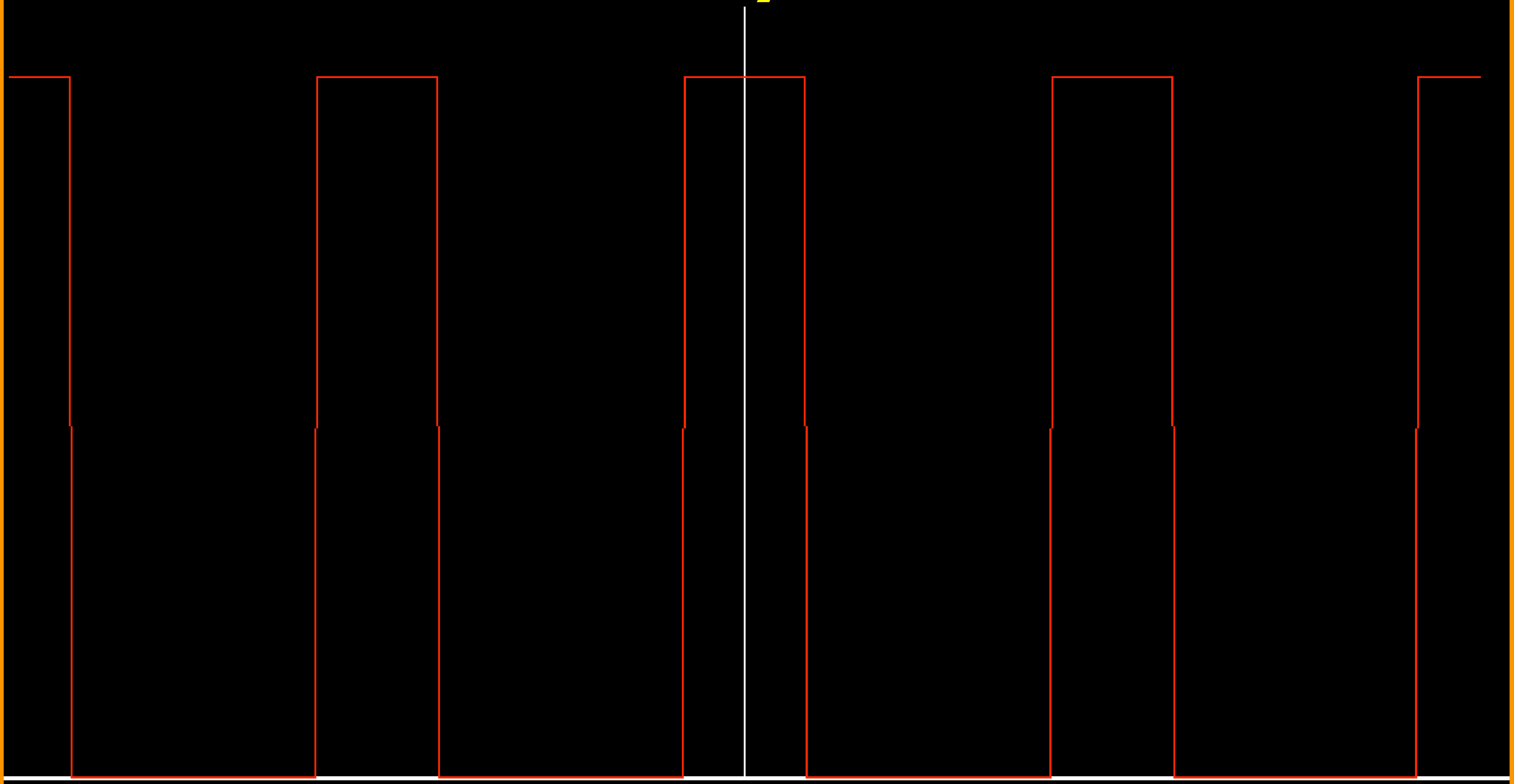
Fourier coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2}}\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

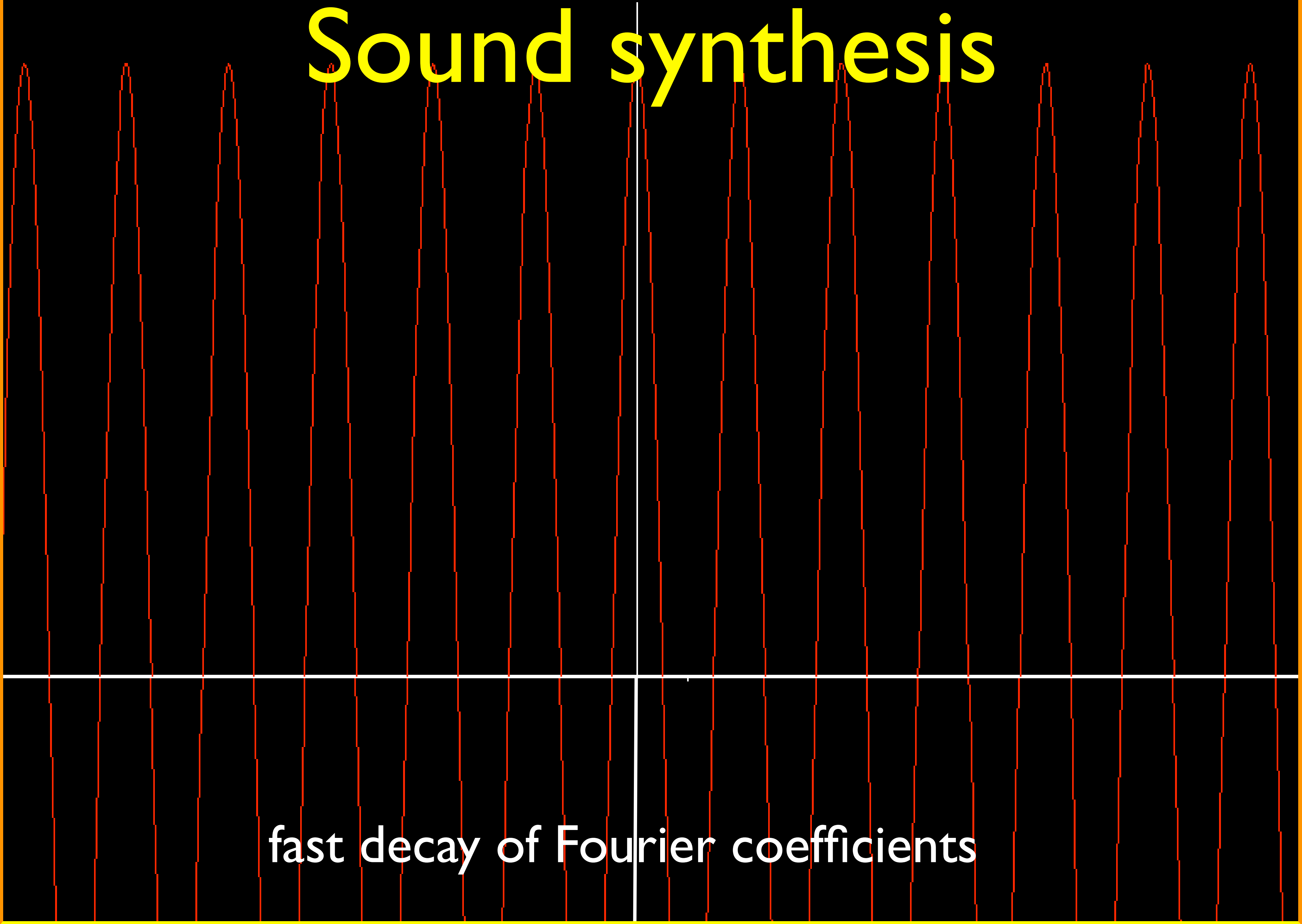
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Sound synthesis



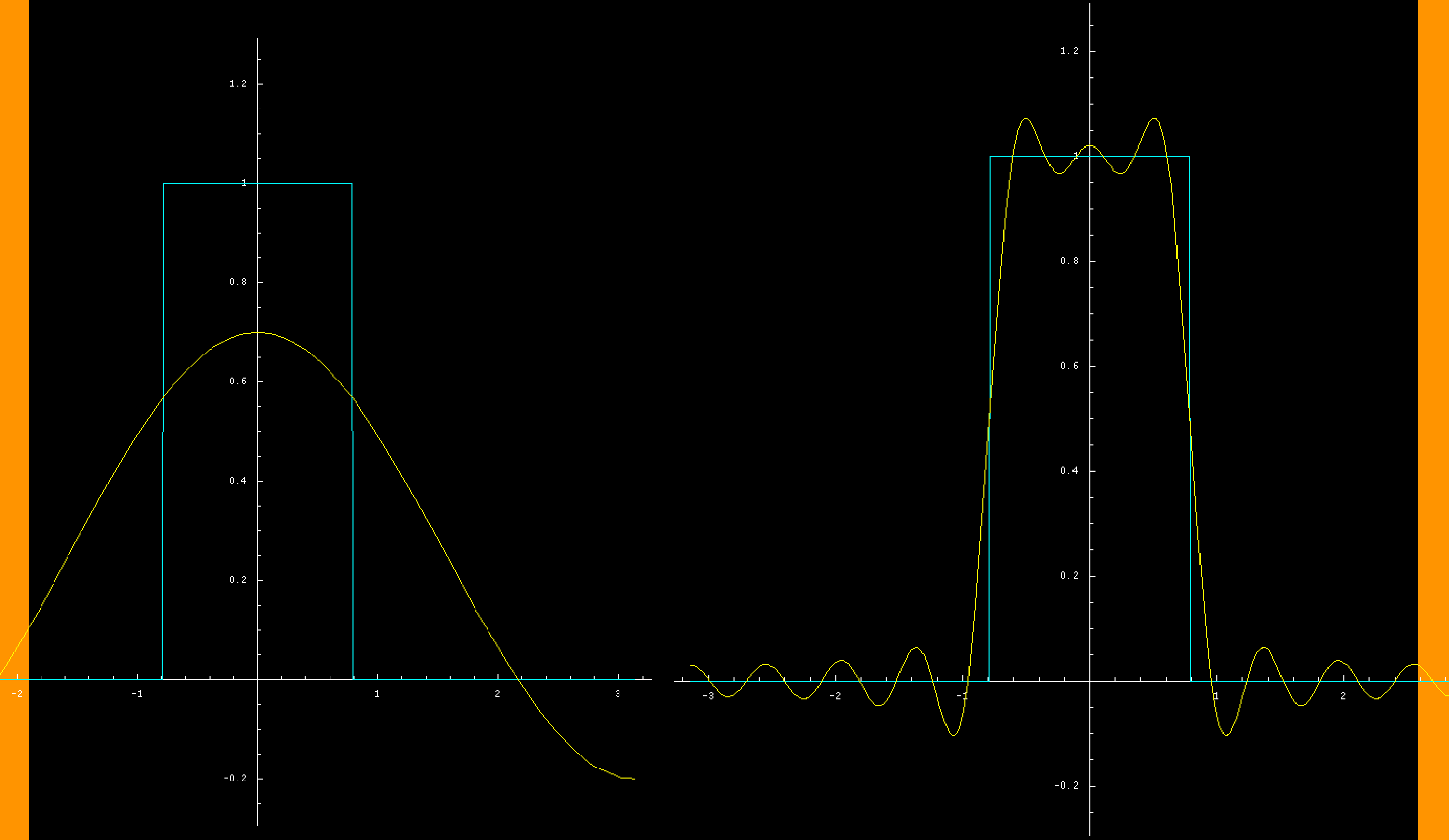
slow decay of Fourier coefficients

Sound synthesis

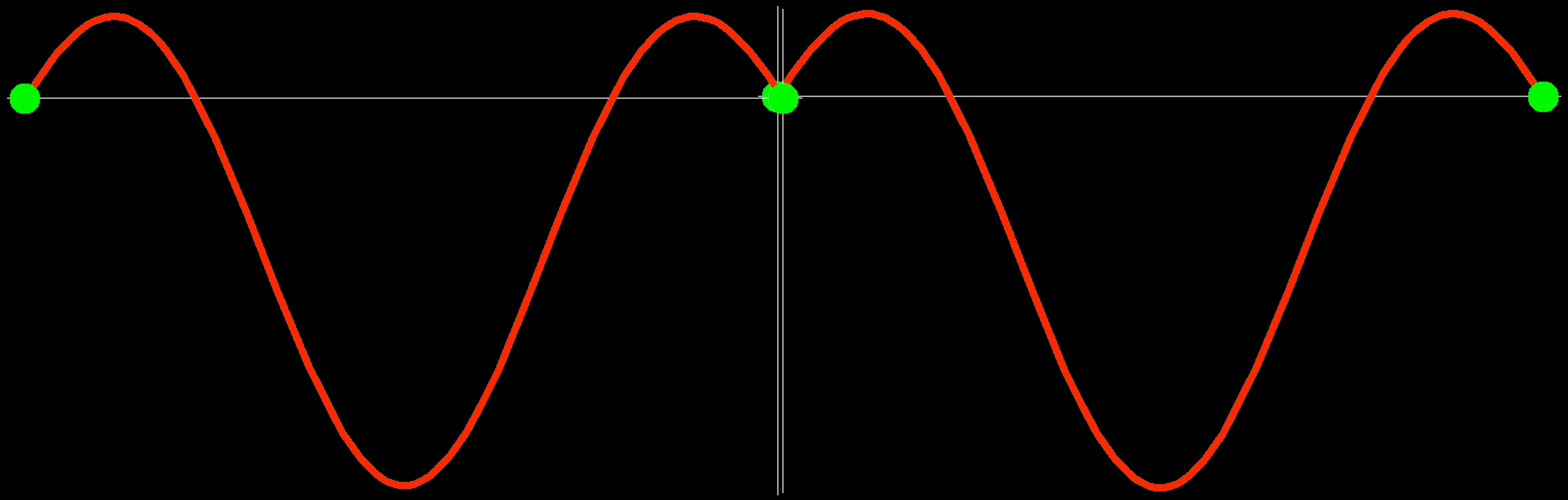


fast decay of Fourier coefficients

Fourier approximation

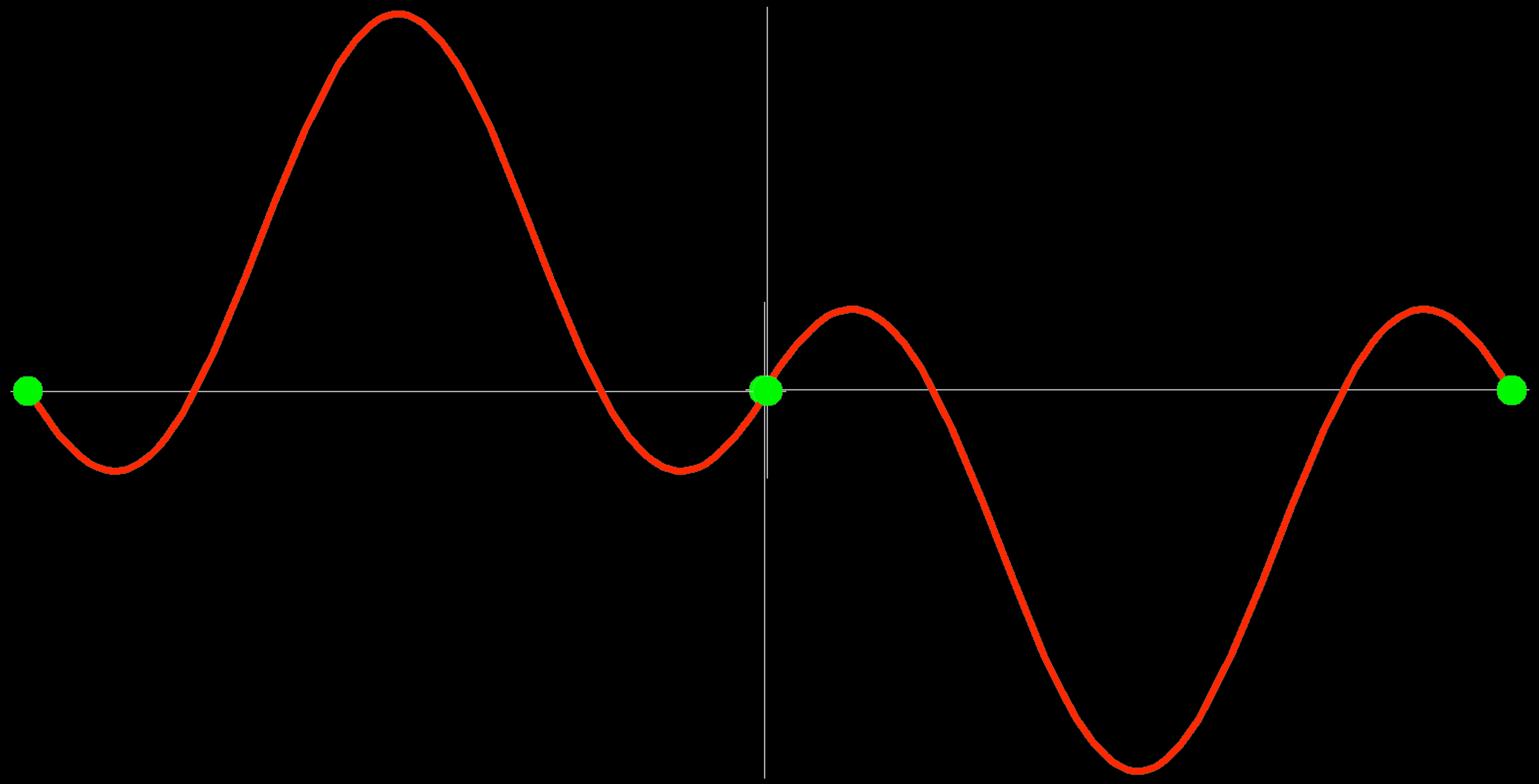


Even Functions



cos- series

Odd Functions



sin- series

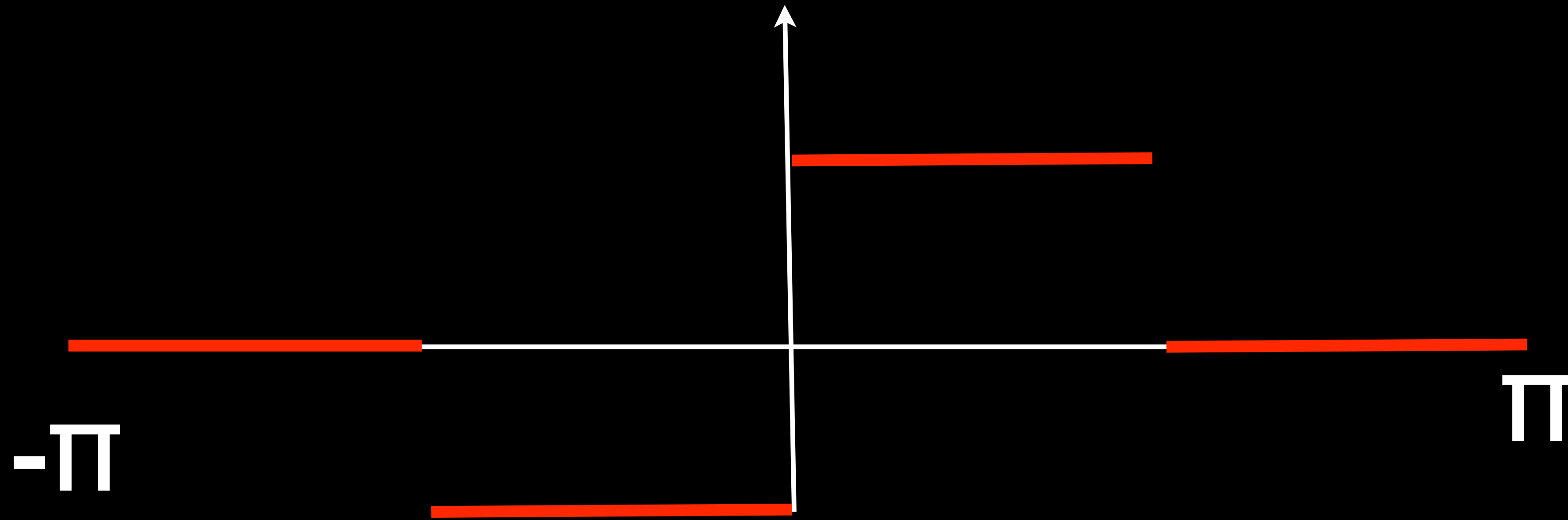

Be
lazy!

If you have an even function, make a cos expansion, if you have an odd function, make a sin expansion. For PDEs we always make sin expansions.

blackboard problem

Find the Fourier series of the function defined by:

$f(x,0) = 1$ if $x<\pi/2$ and $f(x,0)=0$ if $x>\pi/2$.



It is an odd function.

The Fourier coefficients are

$$b_n = \frac{2(1-\cos(n\pi/2))}{n\pi}$$

Parceval Identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$$

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$$

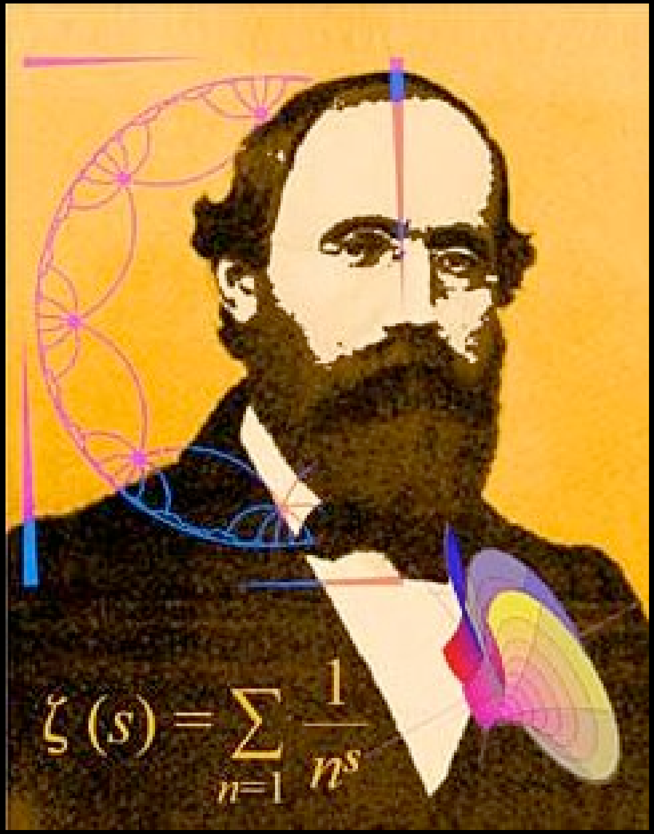
blackboard problem

We verify that

$$1 + \frac{1}{16} + \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}\,.$$

This is the value of the zeta function at s =4.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$



$$x = \sum_{n=1}^{\infty} 2(-1)^{(n+1)} \frac{\sin(nx)}{n}$$

integrate both sides

$$\frac{x^2}{2} = \sum_{n=1}^{\infty} 2(-1)^{n} \frac{\cos(nx)}{n^2} + a_0 \frac{1}{\sqrt{2}}$$

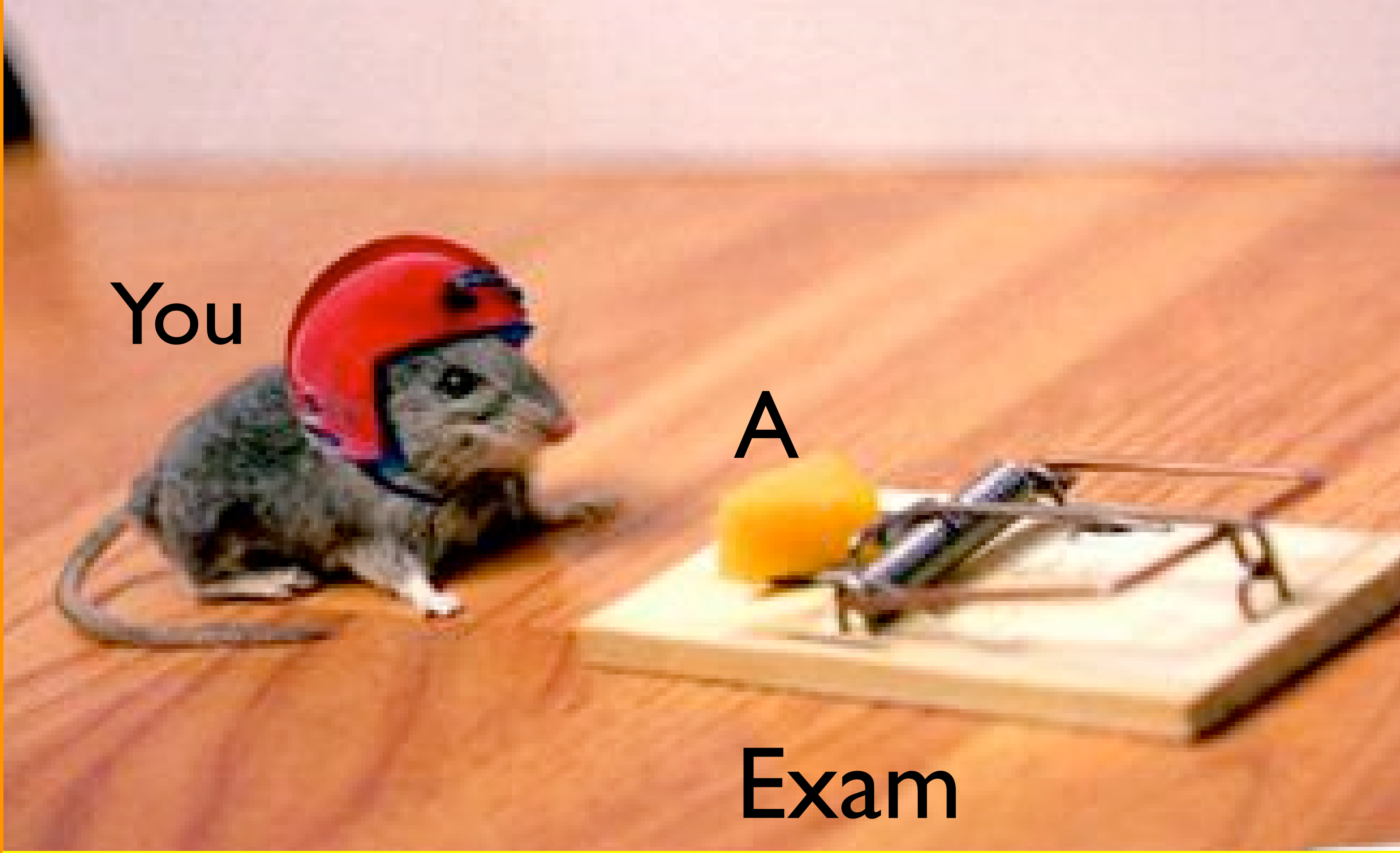
using Perceval and

$$||g||^2 = \langle g, g \rangle = \pi^4/10$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{\pi^4}{10} - \frac{\pi^4}{18}\right) = \frac{2\pi^4}{45}$$

a_0^2

Saftety tips



Separate even and odd parts!

Trigonometric polynomials are already Fourier series i.e. sin(5 x) + cos(3 x)

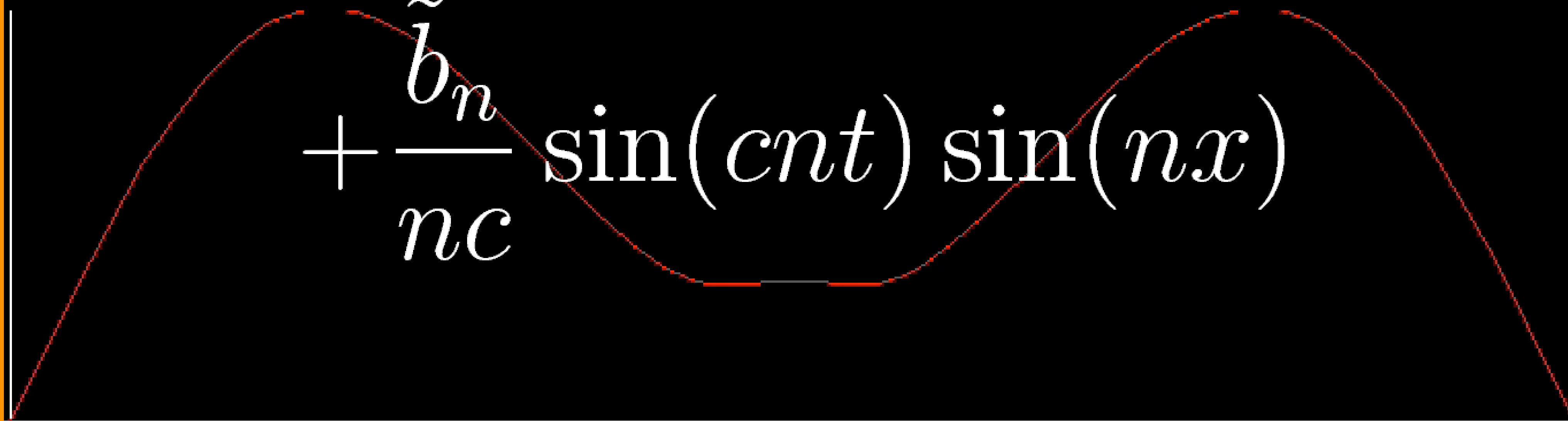
Don’t try to add up the sum. The sum is the final result

Partial differential equations

heat equation

wave equation

$$f(x,t) = \sum_{n \geq 1} b_n \cos(cnt) \sin(nx)$$



$$+ \frac{\tilde{b}_n}{nc} \sin(cnt) \sin(nx)$$

b_n are Fourier coefficients of f(x,0)

$\tilde{b}_n$ are Fourier coefficients of f'(x,0)

Wave evolution

Heat evolution

$$f(x,t) = \sum_{n\geq 1} b_n e^{-n^2 \mu t} \sin(nx)$$

b_n are Fourier coefficients of f(x,0)

blackboard problem

Find the solution of the wave equation

$$u_{tt} = 9\, u_{xx}$$

with initial wave position $f(x,0)=0$ and initial wave velocity $f(x,0) = 1$ if $x<\pi/2$ and $f(x,0)=0$ if $x>\pi/2$.

$$f(x,t) = \sum_{n\geq 1} \frac{b_n}{3n} \sin(3nt)\sin(nx)$$

$$b_n = \frac{2(1-\cos(n\pi/2))}{n\pi}$$


The end