Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. (The actual exam will have more free space). If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

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|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
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| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) (20 points) True or False? No justifications are needed.

**T** F — A linear system with 2 equations and 3 unknowns has either infinitely many or no solutions.

**Solution:**
Geometrically, this is an intersection of two planes. They are either the same plane or intersect in a line.

**T** F — If $S$ is an invertible matrix which contains the vectors $\vec{v}_1, ..., \vec{v}_n$ as columns, then $\vec{v}_1, ..., \vec{v}_n$ is a basis of $\mathbf{R}^n$.

**Solution:**
This is an important property of a basis in $\mathbf{R}^n$. Note that if $A$ is invertible, then it must be a $n \times n$ matrix. Row reduction shows that the image of $A$ is the entire $\mathbf{R}^n$ so that the vectors span. Having no kernel is equivalent that the vectors are linearly independent.

T **F** — If $A, B$ are given $n \times n$ matrices, then the formula $(A-B)(A+B) = A^2 - B^2$ holds.

**Solution:**
Matrix multiplication is not commutative.

**T** F — Suppose $A$ is an $m \times n$ matrix, where $n < m$. If the rank of $A$ is $m$, then there is a vector $y \in \mathbf{R}^m$ for which the system $Ax = y$ has no solutions.

**Solution:**
The matrix has $m$ rows and $n$ columns. The rank can be maximally $n$. The statement that the rank of $A$ is $m$ is not possible. This this was an false assumption, the conclusion is true. You might have come to the right conclusion using other reasoning: there is at least one row for which all entries are 0 entries. After row reduction of the augmented matrix $[A|b]$, we can end up with a last row [0,...,0—1] which means, we have no solution

T **F** — The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is invertible.

**Solution:**
You see immediatly that two rows are parallel. The image is a plane.

**T** F — The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.

**Solution:**
You see that after row reduction all nonzero diagonal entries will be 1, while the zero diagonal entries will stay 0.

T **F** The row reduced echelon form of a $3 \times 3$ matrix of rank 2 is one of the following $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

**Solution:**
We can also have examples, where only the first row has nonzero entries.

T **F** The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a shear.

**Solution:**
A shear has the property that $Av = v$ for some vector $v$. This would mean that $A - 1_2$ has a kernel. But $A - 1 = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$ is invertible.

**T** F For any matrix $A$, one has $\dim(\ker(A)) = \dim(\ker(\mathrm{rref}(A)))$.

**Solution:**
Row reduction does not change the kernel.

T **F** If $\ker(A)$ is included in $\mathrm{im}(A)$, then $A$ is not invertible.

**Solution:**
Also for invertible $A$, the kernel is included in the image.

**T** F There exists an invertible $3 \times 3$ matrix, for which 7 of the 9 entries are $\pi$.

**Solution:**
Example: $\begin{bmatrix} \pi & \pi & \pi \\ \pi & 0 & \pi \\ \pi & \pi & 0 \end{bmatrix}$.

**T** F The dimension of the image of a matrix $A$ is equal to the dimension of the image of the matrix $\mathrm{rref}(A)$.

**Solution:**
While image changes under row reduction, the dimension of the image does not change.

T **F** There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.

**Solution:**
The inverse is invertible too and has therefore rank $n$ also.

**T** F If $A$ and $B$ are $n \times n$ matrices, then $AB$ is invertible if and only if both $A$ and $B$ are invertible.

**Solution:**
Indeed, the inverse is $(AB)^{-1} = B^{-1}A^{-1}$. To see the other direction: if $AB$ is invertible, then $ran(AB) = n$. Since $im(AB)$ is contained in $im(A)$, $dim(im(AB)) \leq dim(im(A))$. Because the dimension of the image is just the rank, $ran(AB) \leq ran(A)$, so $ran(A) = n$, which shows that $A$ is invertible. Then, $B$ is invertible since $B$ can be written as $A^{-1}(AB)$, which is a product of two invertible matrices.

T **F** There exist matrices $A, B$ such that $A$ has rank 4 and $B$ has rank 7 and $AB$ has rank 5.

**Solution:**
The rank of $AB$ is smaller or equal then the rank of $A$ because the image of $AB$ is contained in the image of $A$.

**T** F There exist matrices $A, B$ such that $A$ has rank 2 and $B$ has rank 7 and $AB$ has rank 1.

**Solution:**
Indeed this is possible even for diagonal matrices.

**T** F If for an invertible matrix $A$ one has $A^2 = A$, then $A = I_2$.

**Solution:**
Multiply the equation $A^2 = A$ with $A^{-1}$.

T **F** If an invertible matrix $A$ satisfies $A^2 = 1$, then $A = I_2$ or $A = -I_2$.

**Solution:**
You can have $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for example.

**T** F The matrix $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$ is invertible for every real number $c$.

**Solution:**
The matrix is invertible as long as the rank is 2. Since we know the image is spanned by the columns, we just need to determine if the columns are scalar multiples of each other. If so, then we must have

$$\begin{bmatrix} c-1 \\ 2 \end{bmatrix} = (1-c)\begin{bmatrix} -1 \\ c+1 \end{bmatrix}.$$

This means that $2 = 1 - c^2$, or $c^2 = -1$, which is impossible. Therefore, the columns are linearly independent, and the matrix is invertible.

| T | F |
|---|---|

For $2 \times 2$ matrices $A$ and $B$, if $AB = 0$, then either $A = 0$ or $B = 0$.

**Solution:**
Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

| T | F |
|---|---|

The plane $x + y - z = 1$ is a linear subspace of three dimensional space.

**Solution:**
The plane does not contain the point $(0, 0, 0)$.

| T | F |
|---|---|

If $T$ is a rotation in space with an angle $\pi/6$ around the $z$ axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

**Solution:**
The vector $e_3$ has the property that $S(e_3) = 0$.

Problem 2) (10 points)

Match each of matrices with one of the geometric descriptions below. You don't have to give explanations.

| Matrix | Enter A-H here. |
|---|---|
| a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | |
| b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | |
| c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | |
| d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | |

| Matrix | Enter A-H here. |
|---|---|
| e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ | |
| f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | |
| g) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | |
| h) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | |

A) Shear along a plane.
B) Projection onto a plane.
C) Rotation around an axes.
D) Reflection at a point.
E) Projection onto a line.
F) Reflection at a plane.
G) Reflection at a line.
H) Identity transformation.

**Solution:**
a) = B)
b) = C)
c) = D)
d) = F)
e) = A)
f) = H)
g) = E)
h) = G)

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.

b) What is the length of the vector $\vec{v} = A^{100}e_1$, where $e_1$ is the first basis vector?

c) In which direction does the vector $\vec{v}$ point?

d) Find a matrix $B$ such that $B^2 = A$.

**Solution:**
a) The matrix is a rotation dilation matrix, a rotation by $\pi/4$ and scaling by $\sqrt{2}$.
b) $A^{100}$ is a composition of a scaling by a factor $\sqrt{2}^{100} = 2^{50}$ and rotation by $\pi$. So, $A^{100} = \begin{bmatrix} -2^{50} & 0 \\ 0 & -2^{50} \end{bmatrix}$. The answer is $2^{50}$.
c) It points to $-e_1$: after each 8 rotations, we are back to the initial position, so also after 96 rotations. The additional 4 rotations turn the vector to $-e_1$
d) A rotation by angle $\pi/8$ and scaling $2^{1/4}$ gives a rotation-dilation matrix with $a = 2^{1/4}\cos(\pi/8), b = 2^{1/4}\sin(\pi/8)$. The matrix is

$$A = 2^{1/4} \begin{bmatrix} \cos(\pi/8) & -\sin(\pi/8) \\ \sin(\pi/8) & \cos(\pi/8) \end{bmatrix} .$$

Problem 4) (10 points)

Let $A$ be a $3\times 3$ matrix such that $A^2 = 0$. That is, the product of $A$ with itself is the zero matrix.

a) Verify that $\mathrm{Im}(A)$ is a subspace of $\ker(A)$.

b) Can $\mathrm{ran}(A) = 2$? If yes, give an example.

c) Can $\mathrm{ran}(A) = 1$? If yes, give an example.

d) Can $\mathrm{ran}(A) = 0$? If yes, give an example.

**Solution:**
a) If $y$ is in the image, then $y = A(x)$ and $A(y) = A^2x = 0$.
b) No: If $\dim(\mathrm{ran}(A)) = 2$, then $\dim(\ker(A)) = 1$ and $\dim(\ker(A^2))$ has maximal 2 dimensions so that the rank of $A^2$ would be at least 1. A direct proof: If the rank of $A$ is 2, then $dim(imA) = 2$. By the rank - nullity theorem, $dim(ker(A)) = 1$. On the other hand, by part (a), $im(A)$ is a subspace of $ker(A)$, so $dim(im(A))$ cannot be bigger than $dim(ker(A))$.
c) Yes: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
d) Yes: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 5) (10 points)

Let $b, c$ be arbitrary numbers. Consider the matrix $A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}$.

a) Find $\mathrm{rref}(A)$ and find a basis for the kernel and the image of $A$.

b) For which $b, c$ is the kernel one dimensional?

c) Can the kernel be two dimensional?

**Solution:**
a) $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & -b \\ 0 & 0 & 0 \end{bmatrix}$. We see that the first two columns are pivot columns. The first two columns of $A$ form a basis of the image. To find a basis for the kernel, introduce a free variable $z = t$ for the last (redundant) column. We get $x - ct = 0, y - bt = 0, z = t$ which gives $(x, y, z) = (ct, bt, t) = (c, b, 1)t$. So $\begin{bmatrix} c \\ b \\ 1 \end{bmatrix}$ is a basis for the kernel.
b) The kernel is always one-dimensional.
c) No, since we know from part (b) that the kernel is always one-dimensional.

Problem 6) (10 points)

Consider the matrix $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$.

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form rref$(A)$ of $A$. Do only one elementary operations at each step.

b) Find the rank of $A$.

c) Find a basis for the image of $A$.

d) Find a basis for the kernel of $A$.

**Solution:**

a) We end up with $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) The rank is 2, then number of pivot columns.

c) A basis of the image are the first two column vectors in $A$.

d) A basis of the kernel is $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 7) (10 points)

Let $A$ be a $2 \times 2$ matrix and $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. We know that $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $A^{2003}$.

Hint. Write $B = (I_2 + C)$, note that $C^2 = 0$ and remember $(1+x)^n = 1 + nx + .. + x^n$.

**Solution:**

$B^{2003} = (1 + 2003C)$ so that $A^{2003} = S(1 + 2003\ C)S^{-1} = 1 + 2003\ (SCS^{-1})$. Because $S^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ and $2003SCS^{-1} = 2003 \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$ we have - would have been easier to add up 2000 year ago... - $A^{2003} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2003 \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} -12017 & 8012 \\ -18027 & 12019 \end{bmatrix}$.

Problem 8) (10 points)

Let $A$ be a $5 \times 5$ matrix. Suppose a finite number of elementary row operations reduces $A$ to the following matrix $B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

a) Find a basis of the kernel of $A$.

b) Suppose the elementary row operations used in reducing $A$ to $B$ are the following:

i) Add row 2 to row 3.
ii) Swap row 2 and row 4.
iii) Multiple row 4 by 1/2.
iv) Subtract row 1 from row 5.

Find a basis of the image of $A$.

**Solution:**

a) The matrix $B$ and the matrix $A$ have the same reduced row echelon form.

rref$(A)$ = rref$(B)$ = $B$ = $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. If we call the variables $x_1, \dots, x_5$, this matrix corresponds to the equations $x_1 - x_4 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_5 = 0$. $x_4$ is a free variable and can have any value $t$, so the elements of the kernel are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis of ker $A$.

b) We reverse the steps:
iv inverse) Add row 1 to row 5
iii inverse) multiply row 4 by 2
ii inverse) swap row 2 and row 4
i inverse) subtract row 2 from row 3

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Pick columns 1,2,3,5 of the last matrix.

Problem 9) (10 points)

a) Find a basis for the plane $x + 2y + z = 0$ in $\mathbf{R}^3$.

b) Find a $3 \times 3$ matrix which represents (with respect to the standard basis) a linear transformation with image the plane $x + 2y + z = 0$ and with the kernel the line $x = y = z$.

**Solution:**

a) The solution is $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

b) The image of any matrix is spanned by the columns of the matrix. Since the plane $x + 2y + 3z = 0$ is spanned by the vectors $\vec{v}_1$ and $\vec{v}_2$ from part (a), let's use $\vec{v}_1$ and $\vec{v}_2$ as the first two columns of our matrix. Then, our matrix looks like $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ for some vector $\vec{v}_3$. Since we want the kernel of this matrix to be spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we need

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, we want $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$, so $\vec{v}_3 = -\vec{v}_1 - \vec{v}_2$. Thus, the matrix is

$$\begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$