FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** $f(x,t)$. While the governing equation for a vector was an ordinary differential equation (ODE), the describing equation will now be a **partial differential equation** (PDE). The function $f(x,t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems can be understood using orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.
The homework to this lecture is at the end of this 2 page handout.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

> Example: for $f(x,t) = \cos(x + 4t^2)$, we have
> - $f_x(x,t) = -\sin(x + 4t^2)$
> - $f_t(x,t) = -8t\sin(x + 4t^2)$.
> - $f_{xx}(x,t) = -\cos(x + 4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,y)$ or $f_t(x,y)$ in this handout. This is an official abbreviation in the scientific literature.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which different partial derivatives occur.

> - $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x - t)$.
> - $f_{tt}(x,t) - f_{xx}(x,t) = 0$ has a solution $f(x,t) = \sin(x - t) + \sin(x + t)$.

THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal bar $[0,\pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x,t)$ at x. The function $f(x,t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x,t)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d}{dt}f = \mu D^2 f$$

We will solve the problem in the same way as we solved

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

where A is a matrix - by diagonalization.

We use that the Fourier basis is just the diagonalization: $D^2\cos(nx) = -n^2\cos(nx)$ and $D^2\sin(nx) = -n^2\sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue n^2. By a symmetry trick, we can focus on sin-series during the entire discussion.

SOLVING THE HEAT EQUATION WITH FOURIER THEORY. The heat equation $f_t(x,t) = \mu f_{xx}(x,t)$ with smooth $f(x,0) = f(x), f(0,0) = f(\pi,t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2\mu t} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Proof: With the initial condition $f(x) = \sin(nx)$, we have the evolution $f(x,t) = e^{-\mu n^2 t}\sin(nx)$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ then $f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t}\sin(nx)$.

A SYMMETRY TRICK. Given a function f on $[0,\pi]$ which is zero on both ends. We can extend the function to an odd function on $[-\pi,\pi]$. We do this because of laziness:

> The Fourier series of an odd function is a pure sin-series. The Fourier coefficients are $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$ (note the factor 2 and the integration on $[0,\pi]$ as you have proven in a homework.)
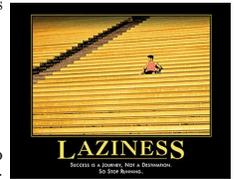
Besides the demotivators: "Success is a journey, not a destination. So stop running", there is an other good argument for laziness, namely **efficiency**: "We want to compute as few Fourier coefficients as possible".
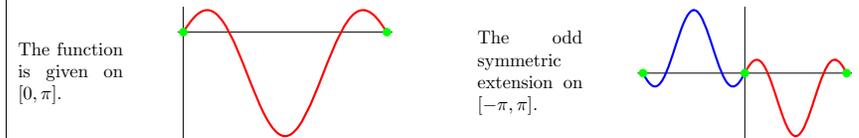
The function is given on $[0,\pi]$.

The odd symmetric extension on $[-\pi,\pi]$.

EXAMPLE. Assume the initial temperature distribution is a saw-tooth function which has on the interval $[0,\pi/2]$ slope $\pi/2$, on the interval $[\pi/2,\pi]$ the slope $-\pi/2$. Let us compute the sin-Fourier coefficients of this function.
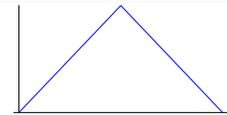
The sin-Fourier coefficients are $b_n = \frac{4}{n^2\pi}(-1)^{(n-1)/2}$ for odd n and 0 for even n. The solution is

$$f(x,t) = \sum_{n}^{\infty} b_n e^{-\mu n^2 t}\sin(nx)\ .$$

The exponential term containing the time makes the function $f(x,t)$ converge to 0: The body cools. The higher frequencies are damped faster: "smaller disturbances are smoothed out faster."

VISUALIZATION. We can just plot the graph of the function $f(x,t)$ or plot the temperature distribution for different times t.

$f(x,0)$ $\qquad$ $f(x,1)$ $\qquad$ $f(x,2)$ $\qquad$ $f(x,3)$ $\qquad$ $f(x,4)$
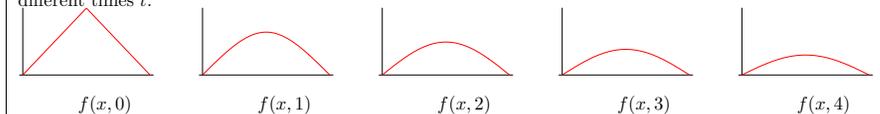
THE WAVE EQUATION. The position of a string $f(x,t)$ at time t on $[0,\pi]$ satisfies the **wave equation**

$$f_{tt}(t,x) = c^2 f_{xx}(t,x)$$

where c is a constant. As we will see, c is the **speed** of the waves.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d^2}{dt^2} f = c^2 D^2 f$$

We will solve the problem in the same way as we solved

$$\frac{d^2}{dt^2} \vec{x} = A\vec{x}$$

If A is diagonal, then every basis vector x satisfies an equation of the form $\frac{d^2}{dt^2}x = -c^2 x$ which has the solution $x(t) = x(0)\cos(ct) + x(t)\sin(ct)/c$.

SOLVING THE WAVE EQUATION WITH FOURIER THEORY. The wave equation $f_{tt} = c^2 f_{xx}$ with $f(x,0) = f(x), f_t(x,0) = g(x), f(0,t) = f(\pi,t) = 0$ has the solution
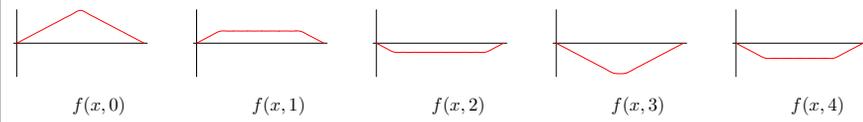
$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$
$$b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx$$

Proof: With $f(x) = \sin(nx), g(x) = 0$, the solution is $f(x,t) = \cos(ct)\sin(nx)$. With $f(x) = 0, g(x) = \sin(nx)$, the solution is $f(x,t) = \frac{1}{c}\sin(ct)\sin(nx)$. For $f(x) = \sum_n a_n \sin(nx)$ and $g(x) = \sum_n b_n \sin(nx)$, we get the formula by summation of those two solutions.


VISUALIZATION. We can just plot the graph of the function $f(x,t)$ or plot the string for different times t.

$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x,t)$ is proportional to the kinetic energy at x. Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly either to the right or to the left. If $f_i(t)$ is the **energy** of particles in cell i at time t, then the energy of particles at time $t+1$ is proportional to $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$. This is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.
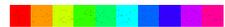
TO THE DERIVATION OF THE WAVE EQUATION. We can model a string by n discrete particles linked by strings. Assume that the particles can move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i+1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$ which is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.
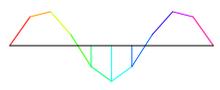
OVERVIEW: The heat and wave equation can be solved like ordinary differential equations:

Ordinary differential equations	Partial differential equations
$$x_t(t) = Ax(t)$$ $$x_{tt}(t) = Ax(t)$$	$$f_t(t,x) = f_{xx}(t,x)$$ $$f_{tt}(t,x) = f_{xx}(t,x)$$
Diagonalizing A leads for eigenvectors $\vec{v}$ $$Av = -c^2 v$$ to the differential equations $$v_t = -c^2 v$$ $$v_{tt} = -c^2 v$$ which are solved by $$v(t) = e^{-c^2 t}v(0)$$ $$v(t) = v(0)\cos(ct) + v_t(0)\sin(ct)/c$$	Diagonalizing $T = D^2$ with eigenfunctions $f(x) = \sin(nx)$ $$Tf = -n^2 f$$ leads to the differential equations $$f_t(x,t) = -n^2 f(x,t)$$ $$f_{tt}(x,t) = -n^2 f(x,t)$$ which are solved by $$f(x,t) = f(x,0)e^{-n^2 t}$$ $$f(x,t) = f(x,0)\cos(nt) + f_t(x,0)\sin(nt)/n$$

NOTATION:

f function on $[-\pi,\pi]$ smooth or piecewise smooth.
t time variable
x space variable
D the partial differential operator $Df(x) = f'(x) = d/dx f(x)$.
T linear transformation, like $Tf = D^2 f = f''$.
c speed of the wave.

$Tf = \lambda f$ Eigenvalue equation analogously to $Av = \lambda v$.
f_t partial derivative of $f(x,t)$ with respect to time t.
f_x partial derivative of $f(x,t)$ with respect to space x.
f_{xx} second partial derivative of f twice with respect to space x.
μ heat conductivity
$f(x) = -f(-x)$ odd function, has sin Fourier series

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1) Solve the heat equation $f_t = \mu f_{xx}$ on $[0,\pi]$ for the initial condition $f(x,0) = |\sin(3x)|$.

2) We want to see in this exercise how to deal with solutions to the heat equation, where the boundary values are not 0.

a) Verify that for any constants a,b the function $h(x,t) = (b-a)x/\pi + a$ is a solution to the heat equation.

b) Assume we have the problem to describe solutions $f(x,t)$ to the heat equations, where $f(0,t) = a$ and $f(\pi,t) = b$. Show that $f(x,t) - h(x,t)$ is a solution of the heat equation with $f(0,t) = 0$ and $f(\pi,t) = 0$.

c) Solve the heat equation with the initial condition $f(x,0) = f(x) = \sin(3x) + x/\pi$ and satisfying $f(0,t) = 0, f(\pi,t) = 1$ for all times t. This is a situation, when the stick is kept at constant but different temperatures on the both ends.

3) A piano string is fixed at the ends $x = 0$ and $x = \pi$ and initially undisturbed. The piano hammer induces an initial velocity $u_t(x,t) = g(x)$ onto the string, where $g(x) = \sin(2x)$ on the interval $[0,\pi/2]$ and $g(x) = 0$ on $[\pi/2,\pi]$. Find the motion of the string.