

## INNER PRODUCT

Math 21b, Fall 2004

RECALL. With the **dot product** in  $\mathbf{R}^n$ , we were able to define **angles**, **length**, compute **projections** or **reflections**. Especially recall that if  $\vec{w}_1, \dots, \vec{w}_n$  was an orthonormal set of vectors, then

$$\vec{v} = a_1 \vec{w}_1 + \dots + a_n \vec{w}_n$$

with  $a_i = \vec{v} \cdot \vec{w}_i$ . This was the formula for the orthonormal projection in the case of an orthogonal set. We will aim to do the same for functions. But first we need to define a "dot product" for functions.

THE INNER PRODUCT. For piecewise smooth functions on  $[-\pi, \pi]$ , we define the **inner product**

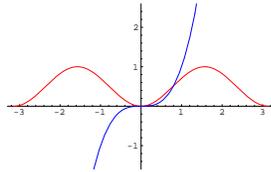
$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$$

It plays the role of the dot product in  $\mathbf{R}^n$ . It has the same properties as the usual dot product: for example,  $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$  or  $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$ .

EXAMPLES.

- $f(x) = x^2$  and  $g(x) = \sqrt{x}$ . Then  $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{3/2} dx = \frac{1}{\pi} x^{5/2} \Big|_{-\pi}^{\pi} = \frac{4}{5} \sqrt{\pi^3}$ .
- $f(x) = \sin^2(x)$ ,  $g(x) = x^3$ . Then  $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x)x^3 dx = \dots$

Hold on with the second example, before integrating. It is always a good idea to look for some symmetry. You can immediately see the answer if you realize that.



ANGLE, LENGTH, DISTANCE, ORTHOGONALITY.

- The **angle**  $\alpha$  between two functions  $f$  and  $g$   $\cos(\alpha) = \frac{\langle f, g \rangle}{\|f\| \|g\|}$
- the **length**  $\|f\|^2 = \langle f, f \rangle$
- and **distance**  $\|f - g\|$  between two functions

are defined in the same way as in finite dimensions. Note that  $\|f\| = 0$  implies that  $f$  is identically 0. Two functions whose distance is zero are identical.

PROPERTIES.

**triangle inequality**  $\|f + g\| \leq \|f\| + \|g\|$ .

**Cauchy-Schwartz inequality**  $|\langle f, g \rangle| \leq \|f\| \|g\|$

**Pythagoras theorem**  $\|f + g\|^2 = \|f\|^2 + \|g\|^2$  for orthogonal  $f, g$ .

EXAMPLE: ANGLE COMPUTATION.

Problem: Find the angle between the functions  $f(t) = t^3$  and  $g(t) = t^4$ .

Answer: The angle is  $90^\circ$ . This can be seen by symmetry. The integral on  $[-\pi, 0]$  is the negative then the integral on  $[0, \pi]$  so that the inner product  $\langle f, g \rangle = 0$ .

EXAMPLE: GRAM SCHMIDT ORTHOGONALIZATION.

**Problem:** Given a two dimensional plane spanned by  $f_1(t) = 1$ ,  $f_2(t) = t^2$ , use Gram-Schmidt orthonormalization to get an orthonormal set.

**Solution:** The function  $g_1(t) = 1/\sqrt{2}$  has length 1. To get an orthonormal function  $g_2(t)$ , we use the formula of the Gram-Schmidt orthogonalization process: first form  $h_2(t) = f_2(t) - \langle f_2(t), g_1(t) \rangle g_1(t)$  then get  $g_2(t) = h_2(t)/\|h_2(t)\|$ .

EXAMPLE: PROJECTION.

**Problem:** Project the function  $f(t) = t$  onto the plane spanned by the functions  $\sin(t)$ ,  $\sin(2t)$ .

**Solution:** Note first that  $\sin(t)$ ,  $\sin(2t)$  form an orthonormal basis in the plane they span. Now  $P(f) = f = \langle t, \sin(t) \rangle \sin(t) + \langle t, \sin(2t) \rangle \sin(2t)$

$$\langle t, \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(t) dt = \frac{-t \cos(t)}{\pi} \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) dt = 2$$

$$\langle t, \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(2t) dt = \frac{-t \cos(2t)}{2\pi} \Big|_{-\pi}^{\pi} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2t) dt = -1$$

The answer is  $P(f) = 2 \sin(t) - \sin(2t)$ . If we would continue this with more  $\sin(kx)$ , we ended up with the Fourier series  $f(t) = 2(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4} \dots)$  More on that on Wednesday.

EXAMPLE: REFLECTION.

**Problem:** Reflect the function  $f(t) = \sin(t)$  at the line spanned by the function  $g(t) = t$ .

**Solution:** Let  $c = \|g\|$ . The projection of  $f$  onto  $g$  is  $h = \langle f, g \rangle g / c^2$ . The reflected vector is  $T(f) = f + 2(h - f) = 2h - f$ . We have computed already  $\langle f, g \rangle = 2$ . With  $c^2 = 2\pi^2/3$ , we have an explicit formula  $T(f) = 2h - f = 2(2t/3)/(2\pi^2) = 6t/\pi^2 - \sin(t)$ .

HOMEWORK.

1. Find the angle between  $f(x) = \cos(x)$  and  $g(x) = x^2$ . (Like in  $\mathbf{R}^n$ , we define the angle between  $f$  and  $g$  to be  $\arccos \frac{\langle f, g \rangle}{\|f\| \|g\|}$  where  $\|f\| = \sqrt{\langle f, f \rangle}$ .)

Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

2. A function on  $[-\pi, \pi]$  is called **even** if  $f(-x) = f(x)$  for all  $x$  and **odd** if  $f(-x) = -f(x)$  for all  $x$ . For example,  $f(x) = \cos x$  is even and  $f(x) = \sin x$  is odd.
  - a) Verify that if  $f, g$  are even functions on  $[-\pi, \pi]$ , their inner product can be computed by  $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) dx$ .
  - b) Verify that if  $f, g$  are odd functions on  $[-\pi, \pi]$ , their inner product can be computed by  $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) dx$ .
  - c) Verify that if  $f$  is an even function on  $[-\pi, \pi]$  and  $g$  is an odd function on  $[-\pi, \pi]$ , then  $\langle f, g \rangle = 0$ .
3. Which of the two functions  $f(x) = \cos(x)$  or  $g(x) = \sin(x)$  is closer to the function  $h(x) = x^2$ ?
4. Determine the projection of the function  $f(x) = x^2$  onto the "plane" spanned by the two orthonormal functions  $g(x) = \cos(x)$  and  $h(x) = \sin(x)$ .

Hint. You have computed the inner product between  $f$  and  $g$  already in problem 1). Think before you compute the inner product between  $f$  and  $h$ . There is no calculation necessary to compute  $\langle f, h \rangle$ .

5. Recall that  $\cos(x)$  and  $\sin(x)$  are orthonormal. Find the length of  $f(x) = a \cos(x) + b \sin(x)$  in terms of  $a$  and  $b$ .