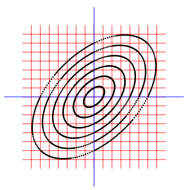
STABILITY — Math 21b, O. Knill

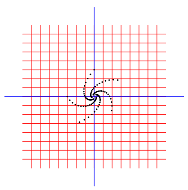
HOMEWORK: 7.6 8,20,42,38*,46*, 8.1 10,24,6*,56* Due: Wednesday or Monday after Thanksgiving.

LINEAR DYNAMICAL SYSTEM. A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, ...$ The vector $x_n = A^n x_0$ describes the situation of the system at **time** n.

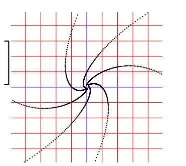
Where does x_n go when time evolves? Can one describe what happens asymptotically when time n goes to infinity?
In the case of the Fibonacci sequence x_n which gives the number of rabbits in a rabbit population at time n, the population grows essentially exponentially. Such a behavior would be called **unstable**. On the other hand, if A is a rotation, then $A^n\vec{v}$ stays bounded which is a type of **stability**. If A is a dilation with a dilation factor < 1, then $A^n\vec{v} \to 0$ for all $\vec{v}$, a thing which we will call **asymptotic stability**. The next pictures show experiments with some **orbits** $A^n\vec{v}$ with different matrices.



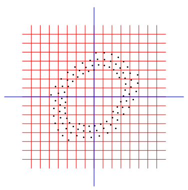
$\begin{bmatrix} 0.99 & -1 \\ 1 & 0 \end{bmatrix}$ stable (not asymptotic)



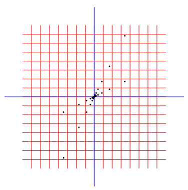
$\begin{bmatrix} 0.54 & -1 \\ 0.95 & 0 \end{bmatrix}$ asymptotic stable



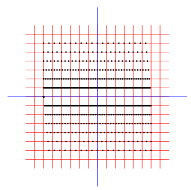
$\begin{bmatrix} 0.99 & -1 \\ 0.99 & 0 \end{bmatrix}$ asymptotic stable



$\begin{bmatrix} 0.54 & -1 \\ 1.01 & 0 \end{bmatrix}$ unstable



$\begin{bmatrix} 2.5 & -1 \\ 1 & 0 \end{bmatrix}$ unstable



$\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$ unstable

ASYMPTOTIC STABILITY. The origin $\vec{0}$ is invariant under a linear map $T(\vec{x}) = A\vec{x}$. It is called **asymptotically stable** if $A^n(\vec{x}) \to \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.

EXAMPLE. Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. Because multiplication wich such a matrix is analogue to the multiplication with a complex number $z = p+iq$, the matrix A^n corresponds to a multiplication with $(p+iq)^n$. Since $|(p+iq)|^n = |p+iq|^n$, the origin is asymptotically stable if and only if $|p+iq| < 1$. Because $\det(A) = |p+iq|^2 = |z|^2$, rotation-dilation matrices A have an asymptotic stable origin if and only if $|\det(A)| < 1$.
Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have eigenvalues $p \pm iq$ and can be diagonalized in the complex.

EXAMPLE. If a matrix A has an eigenvalue $|\lambda| \geq 1$ to an eigenvector $\vec{v}$, then $A^n\vec{v} = \lambda^n\vec{v}$, whose length is $|\lambda^n|$ times the length of $\vec{v}$. So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

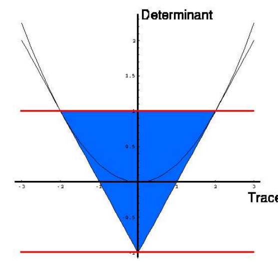
STABILITY. The book also writes "stable" for "asymptotically stable". This is ok to abbreviate. Note however that the commonly used term "stable" also includes linear maps like rotations, reflections or the identity. It is therefore preferable to leave the attribute "asymptotic" in front of "stable".

ROTATIONS. Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i\sin(\phi)$ and are not asymptotically stable.
DILATIONS. Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2. Dilations are asymptotically stable if $|r| < 1$.

CRITERION. A linear dynamical system $x \mapsto Ax$ has an asymptotically stable origin if and only if all its eigenvalues have an absolute value < 1.

PROOF. We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. If $|\lambda_i| < 1$ for all i and all eigenvalues are different, there is an eigenbasis $v_1, \dots, v_n$. Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n(\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \to 0$, there is stability. The proof of the general (nondiagonalizable) case will be accessible later.

THE 2-DIMENSIONAL CASE. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. If $c \neq 0$, the eigenvalues are $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$. If the **discriminant** $(\mathrm{tr}(A)/2)^2 - \det(A)$ is nonnegative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



CRITERION. In two dimensions we have asymptotic stability if and only if $(\mathrm{tr}(A), \det(A))$ is contained in the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \mathrm{tr}(A) - 1$ and $\det(A) = -\mathrm{tr}(A) - 1$.

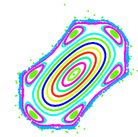
PROOF. Write $T = \mathrm{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1. The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $\boxed{|\mathrm{tr}(A)| - 1 < \det(A) < 1}$.

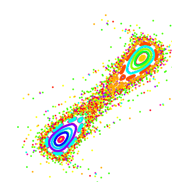
EXAMPLES.
1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has determinant 5/4 and trace 2 and the origin is unstable. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1.
2) A rotation A is never asymptotically stable: $\det(A)1$ and $\mathrm{tr}(A) = 2\cos(\phi)$. Rotations are the upper side of the **stability triangle**.
3) A dilation is asymptotically stable if and only if the scaling factor has norm < 1.
4) If $\det(A) = 1$ and $\mathrm{tr}(A) < 2$ then the eigenvalues are on the unit circle and there is no asymptotic stability.
5) If $\det(A) = -1$ (like for example Fibonacci) there is no asymptotic stability. For $\mathrm{tr}(A) = 0$, we are a corner of the stability triangle and the map is a reflection, which is not asymptotically stable neither.

SOME PROBLEMS.
1) If A is a matrix with asymptotically stable origin, what is the stability of 0 with respect to A^T?
2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{-1}?
3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{100}?

ON THE STABILITY QUESTION.
For general dynamical systems, the question of stability can be very difficult. We deal here only with linear dynamical systems, where the eigenvalues determine everything. For nonlinear systems, the story is not so simple even for simple maps like the Henon map. The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets could get expelled from the solar system (this is a mathematical question because the escape time would be larger than the life time of the sun). For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future ...

A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5$, $g = 2.5$.





The first case is stable (but proving this requires a fancy theory called KAM theory), the second case is unstable (in this case actually the linearization at $\vec{0}$ determines the picture).