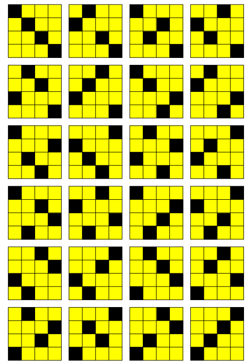
DETERMINANTS I — Math 21b, O. Knill

Section 6.1: 8,18,34,40,44,42*,56*



PERMUTATIONS. A **permutation** of $\{1, 2, \dots, n\}$ is a rearrangement of $\{1, 2, \dots, n\}$. There are $n! = n \cdot (n-1)... \cdot 1$ different permutations of $\{1, 2, \dots, n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest.

EXAMPLE. There are 6 permutations of $\{1, 2, 3\}$: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

PATTERNS AND SIGN. The matrix A with zeros everywhere except $A_{i,\pi(i)} = 1$ is called a permutation matrix or the **pattern** of π. An **inversion** is a pair $k < l$ such that $\sigma(k) < \sigma(l)$. The **sign** of a permutation π, denoted by $(-1)^{\sigma(\pi)}$ is (-1) for an odd number of inversions in the pattern, otherwise, the sign is 1. (To get the sign in the permutations to the right, count the number of pairs of black squares, where the upper square is to the right).

EXAMPLES. $\sigma(1, 2) = 0$, $\sigma(2, 1) = 1$. $\sigma(1, 2, 3) = \sigma(3, 2, 1) = \sigma(2, 3, 1) = 1$. $\sigma(1, 3, 2) = \sigma(3, 2, 1) = \sigma(2, 1, 3) = -1$.

DETERMINANT The **determinant** of a $n \times n$ matrix A is defined as the sum $\boxed{\sum_\pi (-1)^{\sigma(\pi)} A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}}$, where π is a permutation of $\{1, 2, \dots, n\}$ and $\sigma(\pi)$ is its sign.

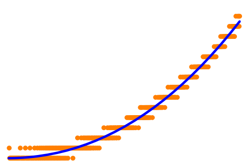
2×2 CASE. The determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $A_{11}A_{12}$, the permutation $(2, 1)$ gives $A_{21}A_{22}$.
If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A. The two vectors form a basis if and only if $\det(A) \neq 0$.

3×3 CASE. The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $aei + bfg + cdh - ceg - fha - bdi$ corresponding to the 6 permutations of $(1, 2, 3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A. The three vectors form a basis if and only if $\det(A) \neq 0$.

EXAMPLE DIAGONAL AND TRIANGULAR MATRICES. The determinant of a diagonal or triangular matrix is the product of the diagonal elements.

EXAMPLE PERMUTATION MATRICES. The determinant of a matrix which has everywhere zeros except $A_{i\pi(j)} = 1$ is just the sign $(-1)^{\sigma(\pi)}$ of the permutation.

HOW FAST CAN WE COMPUTE THE DETERMINANT?.



The cost to find the determinant is the same as for the Gauss-Jordan elimination as we will see below The graph to the left shows some measurements of the time needed for a CAS to calculate the determinant in dependence on the size of the $n \times n$ matrix. The matrix size ranges from n=1 to n=300. We also see a best cubic fit of these data using the least square method from the last lesson. It is the cubic $p(x) = a + bx + cx^2 + dx^3$ which fits best through the 300 data points.

WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- have geometric interpretation as volume
- explicit algebraic expressions for inverting a matrix
- as a natural functional on matrices it appears in formulas in particle or statistical physics
- allow to define orientation in any dimensions
- appear in change of variable formulas in higher dimensional integration.
- proposed alternative concepts are unnatural, hard to teach and and harder to understand
- determinants are fun

TRIANGULAR AND DIAGONAL MATRICES. The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}) = 20$.

PARTITIONED MATRICES. The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}) = 2 \cdot 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i'th column,

$$\boxed{\det([v_1, ..., v, ...v_n]) + \det([v_1, ..., w, ...v_n]) = \det([v_1, ..., v + w, ...v_n])}$$

In general, one has $\boxed{\det([v_1, ..., kv, ...v_n]) = k \det([v_1, ..., v, ...v_n])}$ The same identities hold for rows and follow directly from the original definition of the determimant.

PROPERTIES OF DETERMINANTS.

$\boxed{\det(AB) = \det(A)\det(B)}$ $\boxed{\det(SAS^{-1}) = \det(A)}$ $\boxed{\det(\lambda A) = \lambda^n \det(A)}$

$\boxed{\det(A^{-1}) = \det(A)^{-1}}$ $\boxed{\det(A^T) = \det(A)}$ $\boxed{\det(-A) = (-1)^n \det(A)}$

If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB) = \det(A)\det(B)$, one brings the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.
PROOF OF $\det(A^T) = \det(A)$. The transpose of a pattern is a pattern with the same signature.

PROBLEM. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$.

SOLUTION. Three row transpositions give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 3 & 16 \end{vmatrix}$.

SOLUTION. $\det(A) = 10$, $\det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot$ gogool. This name as well as the gogoolplex $= 10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52}kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ROW REDUCED ECHELON FORM. Determining $\mathrm{rref}(A)$ also determines $\det(A)$.
If A is a matrix and α_i are the factors which are used to scale different rows and s is the number of times, two rows are switched, then $\boxed{\det(A) = (-1)^s \alpha_1 \cdots \alpha_n \det(\mathrm{rref}(A))}$.

INVERTIBILITY. Because of the last formula: $\boxed{\text{A } n \times n \text{ matrix } A \text{ is invertible if and only if } \det(A) \neq 0.}$

THE LAPLACE EXPANSION. (This is the **definition of determinants** of the book.) We compute the determinant of $n \times n$ matrices $A = a_{ij}$. Choose a column i. For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ij} called **minor** which does not contain the i'th column and j'th row. One gets

$$\boxed{\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})}$$

This Laplace expansion just arranges the permutations: listing all permutations of the form $(1, *, ..., *)$ of n elements is the same then listing all permutations of $(2, *, ..., *)$ of $(n-1)$ elements.

ORTHOGONAL MATRICES. Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$ Rotations have determinant 1, reflections have determinant -1.

QR DECOMPOSITION. If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1, the determinant of R is the product of the diagonal elements of R.