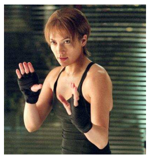
DIMENSION — Math 21b, O. Knill

Homework: Section 3.3: 22,24,32,38,52,36*,56*

LINEAR SPACE. X is a **linear space** if $\vec{0} \in X$ and if X is closed under addition and scalar multiplication.

Examples: $\mathbf{R}^n, X = \ker(A), X = \mathrm{im}(A)$ are linear spaces.

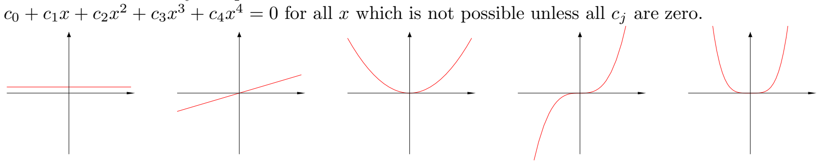
REVIEW BASIS. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$
$\mathcal{B}$ **linear independent**: $c_1\vec{v}_1 + ... + c_n\vec{v}_n = 0$ implies $c_1 = ... = c_n = 0$.
$\mathcal{B}$ **span** X: $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$.
$\mathcal{B}$ **basis**: both linear independent and span.



BASIS: ENOUGH BUT NOT TOO MUCH. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. A basis is, where J.Lo meets A.Hi: Left: J.Lopez in **"Enough"**, right "The man who new **too much**" by A.Hitchcock

AN UNUSUAL EXAMPLE. Let X be the space of polynomials up to degree 4. For example $p(x) = 3x^4 + 2x^3 + x + 5$ is an element in this space. It is straightforward to check that X is a linear space. The "zero vector" is the function $f(x) = 0$ which is zero everywhere. We claim that $e_1(x) = 1, e_2(x) = x, e_3(x) = x^2, e_4(x) = x^3$ and $e_5(x) = x^4$ form a basis in X.

PROOF. The vectors span the space: every polynomial $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ is a sum $f = c_0e_1 + c_1e_2 + c_2e_3 + c_3e_4 + c_4e_5$ of basis elements.
The vectors are linearly independent: a nontrivial relation $0 = c_0e_1 + c_1e_2 + c_2e_3 + c_3e_4 + c_4e_5$ would mean that $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 = 0$ for all x which is not possible unless all c_j are zero.



DIMENSION. The number of elements in a basis of X is independent of the choice of the basis. It is called the **dimension** of X.

UNIQUE REPRESENTATION. $\vec{v}_1, \dots, \vec{v}_n \in X$ **basis** $\Rightarrow$ every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of any line 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

IN THE UNUSUAL EXAMPLE. The set of polynomials of degree ≤ 4 form a linear space of dimension 5.

REVIEW: KERNEL AND IMAGE. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \mathrm{rref}A$. The set of Pivot columns in A form a basis of the image of T, a basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for each non-pivot column.

EXAMPLE. Let X the linear space from above. Define the linear transformation $T(f)(x) = f'(x)$. For example: $T(x^3 + 2x^4) = 3x^2 + 8x^3$. Find a basis for the kernel and image of this transformation.

SOLUTION. Because $T(e_1) = 0, T(e_2) = e_1, T(e_3) = 2e_2, T(e_4) = 3e_3, T(e_5) = 4e_4$, the matrix of T is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is almost in row reduced echelon form. You see that the last four columns are pivot columns. The kernel is spanned by e_1 which corresponds to the constant function $f(x) = 1$. The image is the 4 dimensional space of polynomials of degree ≤ 3.

Mathematicians call a fact a "lemma" if it is used to prove a theorem and if does not deserve the be honored by the name "theorem":

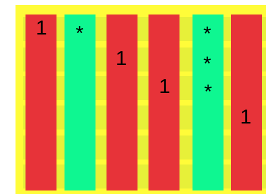
LEMMA. If q vectors $\vec{w}_1, ..., \vec{w}_q$ span X and $\vec{v}_1, ..., \vec{v}_p$ are linearly independent in X, then $p \leq q$.

REASON (an other proof is in the book). Assume $q < p$. Because $\vec{w}_i$ span, each vector $\vec{v}_i$ can be written as $\sum_{j=1}^{q} a_{ij}\vec{w}_j = \vec{v}_i$. After doing Gauss-Jordan elimination of the augmented $(q \times (p+n))$-matrix
$$\left| \begin{array}{ccc|c} a_{11} & \dots & a_{1q} & \vec{w}_1 \\ \dots & \dots & \dots & \dots \\ a_{p1} & \dots & a_{pq} & \vec{w}_q \end{array} \right|$$
which belong to these relations, we end up with a matrix, which contains a last line $\left| \begin{array}{ccc|c} 0 & \dots & 0 & b_1\vec{w}_1 + ... + b_q\vec{w}_q \end{array} \right|$ showing that $b_1\vec{w}_1 + ... + b_q\vec{w}_q = 0$. This nontrivial relation between the the vectors $\vec{w}_i$ is a contradiction to the linear independence. The assumption $q < p$ was absurd.

THEOREM. Given a basis $\mathcal{A} = \{v_1, ..., v_n\}$ and a basis $\mathcal{B} = \{w_1, ..., .w_m\}$ of X, then $m = n$.

PROOF. Because $\mathcal{A}$ spans X and $\mathcal{B}$ is linearly independent, we know that $n \leq m$. Because $\mathcal{B}$ spans X and $\mathcal{A}$ is linearly independent also $m \leq n$ holds. Together, $n \leq m$ and $m \leq n$ implies $n = m$.

DIMENSION OF THE KERNEL. The number of columns in $\mathrm{rref}(A)$ without leading 1's is the **dimension of the kernel** $\dim(\ker(A))$: we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination.
DIMENSION OF THE IMAGE. The number of **leading 1** in $\mathrm{rref}(A)$, the rank of A is the **dimension of the image** $\dim(\mathrm{im}(A))$ because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.



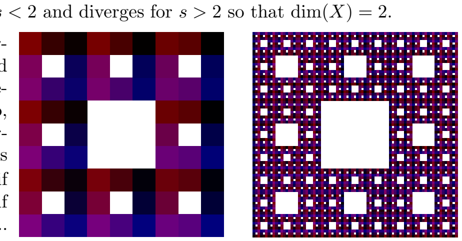
DIMENSION FORMULA: $(A : \mathbf{R}^n \to \mathbf{R}^m)$
$\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$

EXAMPLE: A invertible is equivalant that the dimension of the image is n and that the $\dim(\ker)(A) = 0$.

PROOF. There are n columns. $\dim(\ker(A))$ is the number of columns without leading 1, $\dim(\mathrm{im}(A))$ is the number of columns with leading 1.

EXAMPLE. In the space X of polynomials of degree 4 define $T(f)(x) = f''(x)$. The kernel consists of linear polynomials spanned by e_1, e_2, the image consists of all polynomials of degree ≤ 2. It is spanned by e_3, e_4, e_5. Indeed $\dim(\ker(T)) + \dim(\mathrm{im}(T)) = 2 + 3 = 5 = n$.

FRACTAL DIMENSION. Mathematicians study objects with non-integer dimension since the early 20'th century. The topic became fashion in the 80'ies, when people started to generate fractals on computers. To define fractals, the notion of dimension is extended: define a ***s*-volume of accuracy** r of a bounded set X in $\mathbf{R}^n$ as the infimum of all $h_{s,r}(X) = \sum_{U_j} |U_j|^s$, where U_j are cubes of length $\leq r$ covering X and $|U_j|$ is the length of U_j. The **s-volume** is then defined as the limit $h_s(X)$ of $h_s(X) = h_{s,r}(X)$ when $r \to 0$. The **dimension** is the limiting value s, where $h_s(X)$ jumps from 0 to ∞. Examples:
1) A smooth curve X of length 1 in the plane can be covered with n squares U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n} (1/n)^s = n(1/n)^s$. If $s < 1$, this converges, if $s > 1$ it diverges for $n \to \infty$. So $\dim(X) = 1$.
2) A square X in space of area 1 can be covered with n^2 cubes U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n^2} (1/n)^s = n^2(1/n)^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$ so that $\dim(X) = 2$.
3) The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The $s-$volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k(1/3^k)^s = 8^k/3^{ks}$, which goes to 0 for $k \to \infty$ if $3^{ks} < 8^k$ or $s < d = \log(8)/\log(3)$ and diverges if $s > d$. The dimension is $d = \log(8)/\log(3) = 1.893$..



INFINITE DIMENSIONS. Linear spaces also can have infinite dimensions. An example is the set X of all continuous maps from the real $\mathbf{R}$ to $\mathbf{R}$. It contains all polynomials and because X_n the space of polynomials of degree n with dimension $n+1$ is contained in X, the space X is infinite dimensional. By the way, there are functions like $g(x) = \sum_{n=0}^{\infty} \sin(2^n x)/2^n$ in X which have graphs of fractal dimension > 1 and which are not differentiable at any point x.