Name:

MWF10 Janet Chen

MWF11 Oliver Knill

TTh10 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total:		130

Problem 1) (20 points) True or False? No justifications are needed.

1) [T] [F]   If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

2) [T] [F]   If A and B are 3×3 symmetric matrices, then AB is symmetric.

3) [T] [F]   The solutions of $f'''(x) + f''(x) + f(x) = \sin(x)$ form a linear subspace of all smooth functions.

4) [T] [F]   The initial value problem $f'''(x) + f''(x) + f(x) = \sin(x)$, $f(0) = 0$, $f'(0) = 0$ has exactly one solution.

5) [T] [F]   Every real 3×3 matrix having $1 + i$ as an eigenvalue is diagonalizable over the complex numbers.

6) [T] [F]   If A is a nonzero diagonalizable 4×4 matrix, then A^4 is nonzero.

7) [T] [F]   There exists a real 2×2 matrix A such that $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

8) [T] [F]   There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.

9) [T] [F]   The kernel of the differential operator D^{100} on $C^\infty(\mathbb{R})$ has dimension 100.

10) [T] [F] $Tf(x) = \sin(x)f(x) + f(0) + \int_{-1}^{x} f(y) \, dy$ is a linear transformation on $C^\infty(\mathbb{R})$.

11) [T] [F]   If $S^{-1}AS = B$, then $\operatorname{tr}(A)/\operatorname{tr}(B) = \det(A)/\det(B)$.

12) [T] [F]   If a 3×3 matrix A is invertible, then its rows form a basis of $\mathbb{R}^3$.

13) [T] [F]   A 4×4 orthogonal matrix has always a real eigenvalue.

14) [T] [F]   If A is orthogonal and B satisfies $B^2 = 1$ then AB has determimant 1 or -1.

15) [T] [F]   If $\frac{d}{dt}\vec{x} = A\vec{x}$ has an asymptotically stable origin then $\frac{d}{dt}\vec{x} = -A\vec{x}$ has an asymptotically stable origin.

16) [T] [F]   If $\frac{d}{dt}x = Ax$ has an asymptotically stable origin, then the differential equation $\frac{d}{dt}x = Ax + (x \cdot x)x$ has an asymptotically stable origin.

17) [T] [F]   The transformation on $C^\infty(\mathbb{R})$ given by $T(f)(t) = t + f(t)$ is linear.

18) [T] [F] $\vec{0}$ is a stable equilibrium for the discrete dynamical system $\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$.

19) [T] [F]   If A is an arbitrary 4×4 matrix, then A and A^T are similar.

20) [T] [F]   If A is an invertible 4×4 matrix, then the unique least squares solution to $Ax = b$ is $A^{-1}b$.

[   ] Total

Match the following objects with the correct description. Every equation matches exactly one description.

a) $\frac{d}{dt}x = 3x - 5y$, $\frac{d}{dt}y = 2x - 3y$
b) $f_t = f_{xx} + f_{yy}$.
c) $D^2 f(x) + Df(x) - f(x) = \sin(x)$
d) $\frac{d}{dt}x = 3x^3 - 5y$, $\frac{d}{dt}y = x^2 + y^2 + 2$
e) $\frac{d}{dt}x + 3x = 0$.

i) An inhomogenous linear ordinary differential equation.
ii) A partial differential equation.
iii) A linear ordinary differential equation with two variables.
iv) A homogeneous one-dimensional first order linear ordinary differential equation.
v) A nonlinear ordinary differential equation.

Define $A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix}$.

a) Find rref(A), the reduced row echelon form of A.
b) Find a bases for ker(A) and im(A).
c) Find an orthonormal basis for ker(A).

d) Verify that $\vec{v} \in$ ker(A), where $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$.

e) Express $\vec{v}$ in terms of your orthonormal basis for ker(A).

Find all solutions to the differential equation

$$f''(t) - 2f'(t) + f(t) = 4e^{3t} .$$

Find the unique solution given the initial conditions $f(0) = 1$ and $f'(0) = 1$.

a) Let $f(x)$ be the function which is 1 on $[\pi/3, 2\pi/3]$ and zero elsewhere on the interval $[0, \pi]$. Write f as a Fourier sin-series.

b) Find the solution to the heat equation $T_t = \mu T_{xx}$ with $T(x, 0) = f(x)$.

c) Find the solution to the wave equation $T_{tt} = c^2 T_{xx}$ with $T(x, 0) = f(x)$ and for which $T_t(x, 0) = 0$ holds for all x.

Problem 6) (10 points)

Find a single 3×3 matrix A for which all of the following properties are true.

a) The kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A.

c) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the image of A.

Problem 7) (10 points)

a) Find all solutions to the differential equation $(D^2 - 3D + 2)f = 60e^{7x}$.
b) Find all solutions to the differential equation $(D^2 - 2D + 1)f = x$.
c) Find all solutions to the differential equation $(D^2 + 1)f = x^2$.

Problem 8) (10 points)

Find the matrix for the rotation in $\mathbb{R}^3$ by $90°$ about the line spanned by $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, in a clockwise direction as viewed when facing the origin from the point $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. You get full credit if you leave the result written as a product of matrices or their inverses.

Problem 9) (10 points)

a) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.
b) Is $\vec{0}$ a stable equilibrium point for the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} \ ?$$

c) Describe, how the solution curves of $\frac{d\vec{x}}{dt} = A\vec{x}$ look like.
d) Is $\vec{0}$ a stable equilibrium for the discrete dynamical system $x_{n+1} = Ax_n$?

Problem 10) (10 points)

Does the system

$$\frac{d\vec{x}}{dt} = B\vec{x}$$

with

$$B = \begin{bmatrix} 0 & -1 & -9 & -9 & -8 \\ 0 & 0 & 0 & -1 & -9 \\ 5 & 0 & 5 & 0 & -5 \\ 1 & 9 & 0 & 0 & 0 \\ 1 & 9 & 9 & 8 & 0 \end{bmatrix}$$

have a stable origin?

Hint No lengthy computations are needed. Especially, no eigenvalues have to be computed. If $\lambda_1, ..., \lambda_5$ are the eigenvalues, can you say something about their sum?

Problem 11) (10 points)

A 4×4 matrix A is called **symplectic** if $AJA^T = J$, where $J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

a) Verify that J itself is symplectic.
b) Show that if A is symplectic, then A is invertible and A^{-1} is symplectic.
c) Check that if both A and B are symplectic, then AB is symplectic.
d) Show that for a symplectic matrix A, one has $\det(A) = 1$ or $\det(A) = -1$.

Problem 12) (10 points)

Find the ellipse $f(x, y) = ax^2 + by^2 - 1 = 0$ which best fits the data $(2, 2), (-1, 1), (-1, -1), (2, -1)$.