Recall: In this homework, we look at the **inner product space** with

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \ dx \ .$$

One can define length, distances or angles in the same way as we have done with the dot product for vectors in R^n. Functions are assumed to be (piecewise) smooth.

Homework for first lesson (inner product spaces)

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos \frac{\langle f, g \rangle}{\|f\| \|g\|}$ where $\|f\| = \sqrt{\langle f, f \rangle}$.)

   Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

   Solution:
   From $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(x) \ dx = \frac{1}{\pi} [2x \cos(x) + (x^2 - 2) \sin(x)] \ |_{-\pi}^{\pi} = -4$ and $\|f\|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) \ dx = 1$, $\|g\|^2 = \langle g, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos(x) \ dx = 2\pi^4/5$, we get $\alpha = \arccos(-10/\pi^4)$.

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2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

   a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) \ dx$.

   b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) \ dx$.

   c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g \rangle = 0$.

Solution:

a) If $f(x) = f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} f(y)g(y)\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

b) If $f(x) = -f(-x)$ and $g(x) = -g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)(-g(y))\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

c) If $f(x) = -f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)g(y)\ (-dy) = -\int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 0\ .$$

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

Solution:
The square of the distance between f and g is $||f - g||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin(x) - x^2)^2\ dx = ||f||^2 + ||g||^2 - 2\langle f, g \rangle$. Since f is even and g is odd, this is $||f||^2 + ||g||^2$ (Pythagoras). The square of the distance between f and h is $||f - h||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(x) - x^2)^2\ dx = ||f||^2 + ||h||^2 - 2\langle f, h \rangle$. Because $2\langle f, h \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)x^2\ dx < 0$, the distance from f and h is smaller then the distance from g to h.

4. Determine the projection of the function $f(x) = x^2$ onto the "plane" spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h \rangle$.

Solution:
From the first problem, we know $\langle f, g \rangle = -4$. The dot product $\langle f, h \rangle$ is zero because f is even and h is odd. Because g and h are perpendicular and have length 1, the projection is $P(f) = \langle f, g \rangle g + \langle f, h \rangle h = -4g = -4\cos(x)$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.

Solution:
$||f||^2 = \langle a\cos(x) + b\sin(x), a\cos(x) + b\sin(x)\rangle = \langle a\cos(x), a\cos(x)\rangle + \langle a\cos(x), b\sin(x)\rangle + \langle b\sin(x), a\cos(x)\rangle + \langle b\sin(x), b\sin(x)\rangle = a^2\langle\cos(x), \cos(x)\rangle + b^2\langle\sin(x), \sin(x)\rangle = a^2 + b^2.$