Section 9.4 Solutions

1. a. Solving for the nullclines, we get

$$\frac{dx}{dt} = x(2-x+y) = 0 \Rightarrow \begin{cases} x = 0 & \text{or} \\ y = x-2 \end{cases}$$

$$\frac{dy}{dt} = y(4-x-y) = 0 \Rightarrow \begin{cases} y = 0 & \text{or} \\ y = 4-x \end{cases}$$

The equilibria occur at the intersections of the nullclines, so in the case the equilibrium points are at $(0,0)$, $(2,0)$, $(0,4)$, and $(3,1)$.

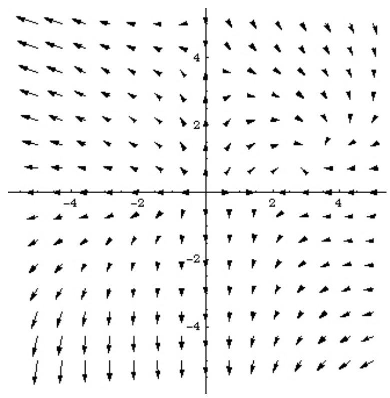

Figure 1: Phase portrait for Exercise 1a.

b. Let $f(x) = x(2-x+y)$ and $g(x) = y(4-x-y)$. Then

$$\mathbf{J} = \begin{Bmatrix} \frac{\partial f}{\partial x}(3,1) & \frac{\partial f}{\partial y}(3,1) \\ \frac{\partial g}{\partial y}(3,1) & \frac{\partial g}{\partial y}(3,1) \end{Bmatrix} = \begin{Bmatrix} -3 & 3 \\ -1 & -1 \end{Bmatrix}$$

c. We see that the eigenvalues of $\mathbf{J}$ at $(3,1)$ are given by $\det(\mathbf{J}(3,1) - \lambda I_2) = \lambda^2 + 4\lambda + 6 = 0 \Rightarrow \lambda = -2 \pm i\sqrt{2}$. Since both eigenvalues have negative real part, the equilibrium is stable.

2. a. Here we need to find the point where $\frac{dx}{dt} = \frac{dy}{dt} = 0$, where x and y are both nonzero. In this case we get

$$\frac{dx}{dt} = x(1-x+ky-k) = 0 \Rightarrow 1-x+ky-k = 0 \Rightarrow x = -k+ky+1$$

$$\frac{dy}{dt} = y(1-y+kx-k) = 0 \Rightarrow 1-y+kx-k = 0$$

$$\Rightarrow 1-y+k(-k+ky+1)-k = 1-k^2+y(k^2-1) = 0$$

$$\Rightarrow y = 1, x = 1$$

So $(a, b) = (1, 1)$ is the equilibrium point we need.

b. The Jacobian at this point is

$$\mathbf{J} = \begin{Bmatrix} \frac{\partial f}{\partial x}(1,1) & \frac{\partial f}{\partial y}(1,1) \\ \frac{\partial g}{\partial y}(1,1) & \frac{\partial g}{\partial y}(1,1) \end{Bmatrix} = \begin{Bmatrix} -1 & k \\ k & -1 \end{Bmatrix}$$

c. The condition for stability is that both eigenvalues have negative real part. In this case we find that $\det(\mathbf{J}(1,1) - \lambda I_2) = \lambda^2 + 2\lambda + (1-k^2) = 0 \Rightarrow \lambda = -1 \pm \sqrt{k^2}$, so that in order for both eigenvalues to have negative real part we must have $|k| < 1$.

3. a. The nullclines are

$$\frac{d\alpha}{dt} = 0 \Rightarrow \omega = 0$$
$$\frac{d\omega}{dt} = 0 \Rightarrow \alpha = n\pi, \text{ for some } n \in \mathbf{Z}$$

The equilibrium points occur at $(\alpha, \omega) = (0, n\pi)$.

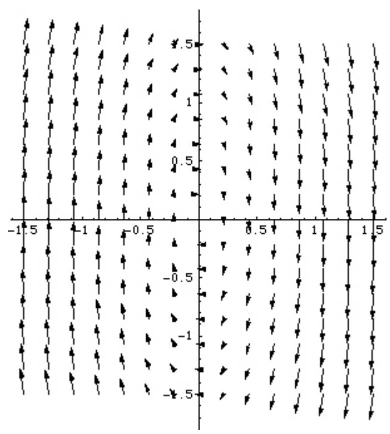

Figure 2: Phase portrait for Exercise 3a.

b. $\mathbf{J} = \begin{Bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{Bmatrix}$. The eigenvalues are $\lambda = \pm i\sqrt{\frac{g}{L}}$. This suggests that the system remains in closed orbits near the equilibrium points, so these points are not stable in the sense in which we have defined them to be.

4. Here the nullclines occur at

$$x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$$
$$xy = 0 \Rightarrow \begin{cases} x = 0, & \text{or} \\ y = 0 \end{cases}$$

The equilibria occur at the points $(\pm 1, 0), (0, \pm 1)$.

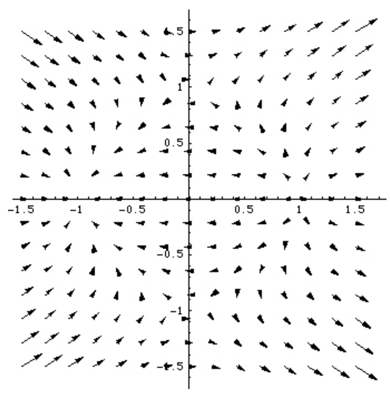

Figure 3: Phase portrait for Exercise 4.

We have that the Jacobian is

$$\mathbf{J} = \begin{Bmatrix} 2x & 2y \\ y & x \end{Bmatrix}$$

So that the eigenvalues are $\det(\mathbf{J} - \lambda I_2) = \lambda^2 - 3x\lambda + 2x^2 - 2y^2 = 0 \Rightarrow \lambda = \frac{1}{2}\left(3x \pm \sqrt{9x^2 - 4(2x^2 - 2y^2}\right) = \frac{1}{2}\left(3x \pm \sqrt{x^2 + 8y^2}\right)$. The stable equilibiria occur when both eigenvalues have negative real part; we see that this occurs at $(-1, 0)$ only.

5. The nullclines of the system are

$$v = 0$$
$$f(u) - v = 0 \Rightarrow v = f(u) = \frac{du}{dt}$$

The only equilibrium point occurs when $(u, v) = (1, 0)$. Note that at this point $f(u) = 0$ and $f'(u) > 0$. Thus we see that

$$\det(\mathbf{J}(1,0) - \lambda I_2) = \begin{vmatrix} 0 - \lambda & 1 \\ f'(u = 1) & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda - f'(u = 1) = 0 \Rightarrow \lambda = \frac{1}{2}(-1 \pm \sqrt{1 + 4f'(1)})$$

Since $\sqrt{1 + 4f'(1)} > \sqrt{5} > 1$, one of the eigenvalues is positive and so this equilibrium is unstable.