If you have questions about posted solutions (like typos, errors, additions, things needing clarification), please email it to math21b@fas.harvard.edu.

Section 10.1 Linear equations.
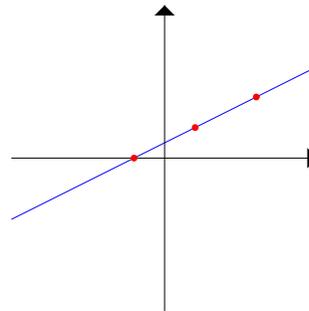10) The equation are

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 7z &= 2 \\ 3x + 7y + 11z &= 8 \end{aligned}$$

Subtracting twice 1) from 2) gives $z = 0$. Subtracting 3 times 1) from 3) gives $y = 5$, then $x = -9$.

12) The solution is the intersection of two lines

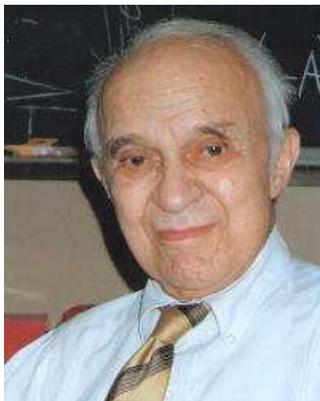$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$$

But these two lines are the same.

20) The Leontief equations are

$$\begin{aligned} a - 0.1b &= 1000 \\ b - 0.2a &= 780 \end{aligned}$$

This system can be solved by adding 5 times the second equation to the first: $4.9b = 4900$ or $b = 1000$ and $a = 1100$.

Remark. Wassily Leontief was born in 1906 in St. Petersburg Russia. He obtained his doctorat in Germany in 1928, was Harvard professor from 1953 to 1975. He received in 1973 the Nobel prize in Economics. He worked last at New York University and died in 1999. He is best known for the development of "input-output" analysis, which has fundamentally influenced economic analysis. In this theory, the structure of the economy is described by a matrix of input-output coefficients which turned out to be very useful: Leontief: "I was not trying to improve the system. I was just concentrating on understanding how it works." (Source: J.S, Landefeld and S.H. McCulla: "Survey of current business", March 1999, p. 9-11).

24) Assume x is the speed of the river and y is the speed of the ship. If L is the distance to travel, then $8000 = 20(x + y)$ and $8000 = 40(x - y)$. We get the system

$$
\begin{aligned}
400 &= x + y \\
200 &= x - y
\end{aligned}
$$

Addition of the two lines gives $2x = 600$ or $x = 300$ (meter per minutes) and $y = 100$ (meter per minutes).

38b) The last equation gives $x_4 = 0$. The second last $x_3 = 2$, the second, $x_2 = -1$, the first finally $x_1 = 1$.
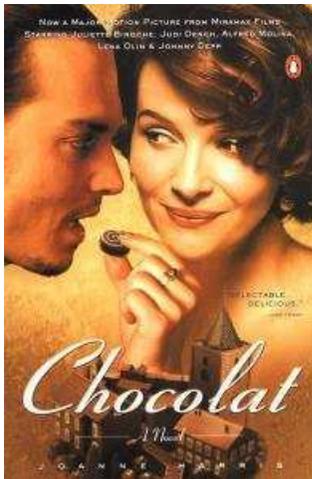
26*) Simplify the system

$$
\begin{aligned}
x + y - z &= 2 \\
x + 2y + z &= 3 \\
x + y + (k^2 - 5)z &= k
\end{aligned}
$$

to get

$$
\begin{aligned}
y + 2z &= 2 \\
(k^2 - 4)z &= k - 2
\end{aligned}
$$

so that $z = (k-1)/(k^2-4)$. For $k \neq 2, -1$ we have exactly one solution. For $k = 2$, the variable z can be anything in the last equation and we have infinitely many solutions. For $k = -2$, the last equation has no solution z. The system is then inconsistent.

36*) If Boris fortune is x and Marinas y, then

$$
\begin{aligned}
x/2 + y &= 2 \\
x + y/2 &= 1
\end{aligned}
$$

Subtracting twice the second from the first equation gives $3/2 x = 0$ and $y = 2$.

Section 10.2 Gauss-Jordan Elimination

6) The system is already in row reduced echelon form. We can chose $x_5 = c$ freely, Then $x_4 = 1 - c$, $x_3 = 2 + 2c$, $x_2 = d$ can be chosen freely and we have $x_1 = 3 - c + 7d$. We see that the solution set has two free parameters.

10) After row reduction of the augmented matrix, we end up with

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can rewrite this as a system of equations

$$\begin{aligned} x_1 - x_4 &= 1 \\ x_2 - 3x_4 &= 2 \\ x_3 - 2x_4 &= -3 \end{aligned}$$

If you fix $x_4 = t$, the other variables are determined: $x_1 = 1 - t, x_2 = 2 - 3t, x_3 = -3 + 2x_4$.

18) Just b) and d) are in row reduced echelon form. In a), we can continue with the third row. In c), the third row should be exchanged with the second.

20) There are 4 possible cases:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where c is a constant.

30) In order that the points are on the cubic curve $y = a + bx + cx^2 + dx^3$, four equations have to be satisfied:

$$\begin{array}{l|l} (0,1) & a = 1 \\ (1,0) & a + b + c + d = 0 \\ (-1,0) & a - b + c - d = 0 \\ (2,-15) & a + 2b + 4c + 8d = -15 \end{array}$$

To do Gauss-Jordan elimination, we start with

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \end{array}\right]$$

and end up with

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, $d = -2, c = -1, b = 2, a = 1$. The cubic polynomial is $y(x) = 1 + 2x - x^2 - 2x^3$.
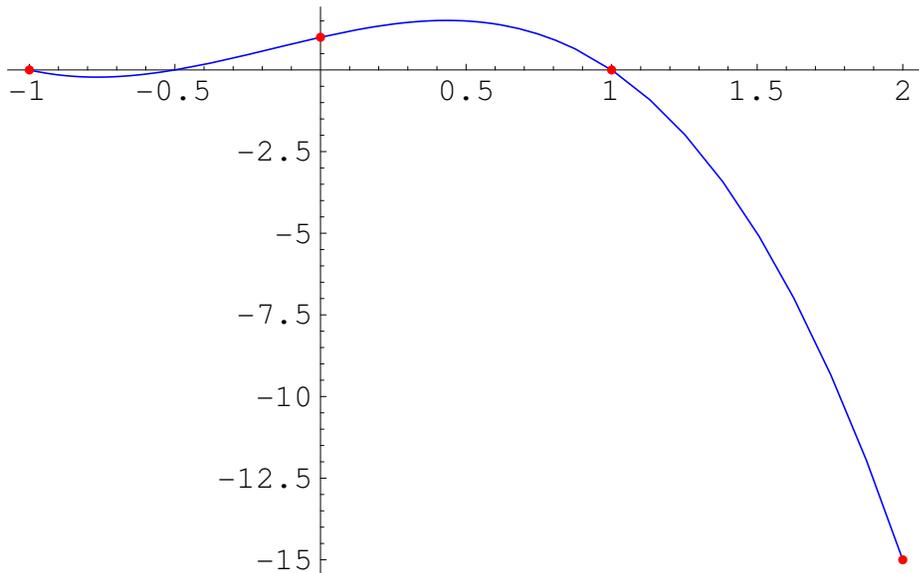
32) The condition $f_i(a_i) = f_{i+1}(a_{i+1})$ assures that the curves meet at the end points. The condition $f_i'(a_i) = f_{i+1}'(a_{i+1})$ assures that the joint curve does not have corners. The condition $f_i''(a_i) = f_{i+1}''(a_{i+1})$ assures even a better fit in that the curvatures of the curves at the end points agree.

Because we have n curves where each curve has 3 parameters, there are $3n$ variables. There are also 3n equations.

38) The demand vectors are

$$v_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \end{bmatrix}$$

The equation $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \vec{b}$ tells that the output vector is the the sum of the consumer demand and suitably scaled demands from the other industries.

Section 10.3 On Solutions of Linear Equations.

4) In order to compute the rank, we do Gauss-Jordan elimination:

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ The rank is the number of leading ones, which is 2 in this case.

14) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 \\ -2+4+4 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. In both cases, we get as a result $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

34) This is an important thing to remember: for any $m \times 3$ matrix, Ae_1 is the first column vector, Ae_2, the second and Ae_3 is the third column vector.

a) $A(\vec{x} + \vec{x}_h) = A\vec{x} + A\vec{x}_h = A\vec{x} + 0 = A\vec{x}$.
b) $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y} = \vec{b} - \vec{b} = 0$.
c) It is a line parallel to the other line passing through the end point of $\vec{x}_1$.

50) There are no solutions because we have a leading one at the end of the last row which would mean $0x + 0y + 0z = 1$.

26*) It can have a unique solution or no solution.

Assume that the system $Ax = b$ is in row reduced echelon form say $A^*x^* = b^*$ and that $B = [A|c]$ goes after row reduction into $B^* = [A^*|c^*]$. There is one solution if and only if the last entry of c^* is zero.

46*) The rank is three. Row reduction produces the identity matrix.

Section 2.1 Linear transformations. (6,14,28,42,44,34*,(24-30)*)

6) Row picture: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are the rows of the matrix, we are looking for. With

$A == \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ one has $A\vec{x} = T(\vec{x})$. Indeed, the transformation is linear.

14) a) In order to see whether a matrix is invertible, we row reduce.

$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ $\begin{bmatrix} 2 & 3 \\ 0 & 15-2k \end{bmatrix}$. The matrix is not invertible, if and only if $2k = 15$.

b) All entries are integers if the solution $\vec{x} = (x, y)$ to the equation $A\vec{x} = \vec{b}$ with integer vector
$\vec{b} = (a, b)$ has integers. If we row reduce the augmented matrix $B = \begin{bmatrix} 2 & 3 & a \\ 5 & k & b \end{bmatrix}$ we end up
with
$\mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & (3b - ak)/(2k - 15) \\ 0 & 1 & (5a - 2b)/(2k - 15) \end{bmatrix}$ which shows that $2k - 15$ must be either a fraction
$1/n$. That means $k = (15 + 1/n)/2$, where n is a nonzero integer.

28) The face gets streched in the y coordinates by a factor 2.

42) a) The image of $\vec{0}$ is $\vec{0}$. The image of the 3 basis vectors e_i are the columns of A. The image
of the other 4 vectors can be obtained by linearity.

b) To find the points which are maped to zero, we must have $-x/2 + y = 0$ and $-x/2 + z = 0$.
Choosing y freely, say s, we have $x = 2s, y = z = s$. So, all the points on the line $\begin{bmatrix} 2s \\ s \\ s \end{bmatrix}$ are
maped to zero.

P.S. We will later call the set of all this vectors the **kernel** of A.

44) Yes, the transformation is linear. The matrix is obtained by applying the map to the
fundamental vectors e_i:
$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

34*) The trick is to draw the images of the basis vectors and put that as the columns of the
matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

24) is a rotation by 90 degrees in the counter clockwise direction,

25) is a scaling (dilation) by a factor 2,

26) is a reflection at the line $x = y$,

27) is a reflection at the y axes,

28) streches by a factor 2 in the y direction.

29) is a reflection at the origin,

30) is a projection onto the y-axes.

Section 2.2 Linear transformations. (2.2: 4,8,10,30,34,47*,50*)

4) It is a rotation dilation. A dilation by a factor $\sqrt{2}$ followed by a rotation by $\pi/2$.

8) A shear parallel to the x axes.

10) Just take the basis vectors and map them under the map $T(\vec{x}) = (\vec{u} \cdot \vec{x})\vec{u}/(25)$. The matrix is $\begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$.

30) It is a projection on the line containing the vector $(1, 2)$.

34) For the transformation which is a projection onto a line, all three basis vectors e_i should be maped into the same vector. That is the case for B only. For a reflection in a line, the map should have the property that $T(T(x)) = x$. This is only the case for transformation E.

47) Define $e_1(t) = R(t)e_1, e_2(t) = R(t)e_2$, where $R(t)$ is the rotation by an angle t. We have $f(t) = T(e_1(t)) \cdot T(e_2(t))$.

a) If $T(x, y) = (ax + by, cx + dy)$, then $f(t) = (a\cos(t) + b\sin(t), c\cos(t) + d\sin(t)) \cdot (-a\sin(t) + b\cos(t), -c\sin(t) + d\cos(t)) = (a^2 + b^2 - c^2 + d^2/2)\sin(2t) + (ab + cd)\cos(2t)$ which is continuous.
b) $f(\pi/2) = -f(0)$ follows from a).
c) This is the intermediate value theorem.
d) If $f(t) = 0$, then the two vectors $T(e_1(t)), T(e_2(t))$ are perpendicular.

This exercice shows that we can turn the coordinate system in such a way that the images of the vectors are perpendicular to each other. If T is noninvertible, then we can turn the coordinate sysetm such that one vector is maped to the zero vector.

50) Assume, we have two perpendicular vectors $\vec{v}_1, \vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are orthogonal. The unit circle can be written as $cos(t)v_1 + \sin(t)v_2$. The image is $\cos(t)w_1 + \cos(t)w_2$.

An other approach would be to verify that transformations like rotations, dilations, diagonal transformations and reflections map ellipses into ellipses and that one can write any transofor-mation as a composition of such transformations.

Section 2.3 Linear transformations. (2.3: 10,20,30,40*,42*)

10)

Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 3 & | & 0 & 1 & 0 \\ 1 & 3 & 6 & | & 0 & 0 & 1 \end{bmatrix}$. gives $\begin{bmatrix} 1 & 0 & 0 & | & 3 & -3 & 1 \\ 0 & 1 & 0 & | & -3 & 5 & 2 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{bmatrix}$.

20) Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 3 & 3 & | & 1 & 0 & 0 \\ 1 & 4 & 8 & | & 0 & 1 & 0 \\ 2 & 7 & 12 & | & 0 & 0 & 1 \end{bmatrix}$, we end up with

$\begin{bmatrix} 1 & 0 & 0 & | & -8 & -15 & 12 \\ 0 & 1 & 0 & | & 4 & 6 & -5 \\ 0 & 0 & 1 & | & -1 & -1 & -1 \end{bmatrix}$.

30) The map is never invertible.

40) If the i'th and j'th row are the same, then the vector $\vec{u} = e_i - e_j$ is maped to the zero vector. This means that the vectors $\vec{v}$ and $\vec{v} + \vec{u}$ are maped to the same point $\vec{z}$. The inverse of that vector $\vec{z}$ is not defined since it both would have to be $\vec{v}$ or $\vec{u}$. In our case, both the vectors $\vec{v}$ and $\vec{u} = \vec{0}$ are maped to $\vec{z}$.

42) Yes, permutation matrices are invertible, because we can invert the permutation. For example, the inverse of the permutation matrix which belongs to the permutation $(1, 2, 3) \to (2, 3, 1)$ is the permutation matrix which belongs to the permutation $(1, 2, 3) \to (3, 1, 2)$.

An other way to see that a permutation matrix is invertible is to row reduce it. It always can be row reduced to the identity matrix by switching rows alone.

Section 2.4 Matrix Multiplication. 2.4: 14,28,48*,72*

14) $BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$, $BD = [6]$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, $DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB = $

$[5, 10, 15]$. If you want to pair matrices with itself, we can also form $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $C^2 = $

$\begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $E^2 = [25]$.

28) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a possibility.

Remark for later: Actually, $A^2 = 0$ implies that A is similar to that transformation (see later). The image of A is contained in the kernel of A and must be one dimensional by the dimension formula (seen later). In a basis, where one of the basis vectors is in the kernel, the transformation looks like that.

40) $AB = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. $A = (AB)B^{-1} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$.

72) Trying $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ gives $A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$. In order that $a = 2a^2$, we must have

$a = 1\sqrt{2}.$

Section 2.4 Matrix Multiplication.

4) $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

14) $BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$, $BD = [6]$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, $DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB =$

$[5, 10, 15]$. If you want to pair matrices with itself, we can also form $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $C^2 =$

$\begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $E^2 = [25]$.

28) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a possibility.

Remark for later: Actually, $A^2 = 0$ implies that A is similar to that transformation (see later). The image of A is contained in the kernel of A and must be one dimensional by the dimension formula (seen later). In a basis, where one of the basis vectors is in the kernel, the transformation looks like that.

40) $AB = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. $A = (AB)B^{-1} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$.

76) a) $A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c) $AB = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}$.

d) $M = A^{-1}BA = \begin{bmatrix} 2/3 & 1/3 & -2/9 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}$.

48) $T : P_2 \to P_2$, $P_1 \to P_3 \to P_0 \to P_1$.
$L : P_0 \to P_0, P_3 \to P_3, P_1 \to P_2 \to P_1$.

a) Inverse rotation T^{-1}. b) Reflection
c) Since $T^3 = I$, we know that T^2 is the inverse rotation T^{-1}.
d) $T \circ L : P_0 \to P_1, P_1 \to P_2, P_2 \to P_3, P_3 \to P_0$.
$L \circ T : P_0 \to P_2, P_1 \to P_3, P_2 \to P_1, P_3 \to P_0$.
e) $P_0 \to P_2 \to P_3 \to P_0, P_1 \to P_1$ is a rotation.

66) A is invertible if and only if A_{11} and A_{22} are both invertible. The inverse is $\begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$.

One can find the inverse by writing out $BA = 1$ as four equations for the four unknown entries of B.

Section 3.1 Image and Kernel

10) We bring the matrix in row reduced echelon form. Starting to clean out the second column $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and end up with $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. If we write down the

system $B \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we get $w = 0, z = s, y + 2s = 0, x - s = 0$, where s is a free vari-

able introduced in the third column, the only one without leading one. The kernel is the line

spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

22) $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The image is spanned by $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$,

the two first columns of A. The kernel is spanned by $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

34) Take the projection on the plane $-x+y+2z = 0$ for example. There are other possibilities.

44) Under a row reduction, the image changes but not the kernel.

For a) we can take as a counter example the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The kernal of A is spanned

by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the image of A is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. the image of $\text{rref}(A)$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b) The kernel does not change under row reduction. Check that for switching rows, scaling rows or adding rows to an other row.

54) If there was no error, then $Hw = 0$ because $H(Mx)) = 0$ if there was no error. The columns of H encode, which of the bits are switched. We can look match $H(M(x + e)))$ with the columns and get the offending bit.

38) a) $Ax = 0$ implies $A^2x = 0$ so that $\ker(A^2)$ contains $\ker(A)$ and more generally, $\ker(A^n)$ contains $\ker(A^{n-1})$. The inclusion can be strict, like with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

b) $\text{im}(A^2)$ is a subset of $\text{im}(A)$ and more generally $\text{im}(A^n)$ is a subset of $\text{im}(A^{n-1})$. The inclusion can be strict, like with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ where the image of A^2 is contained in the one dimensional $\text{im}(A)$.

48)
a) $w = Av$ and $Aw = A^2v = Av = w$ so that $Aw = w$.
b) If $\text{rank}(A) = 2$ then $A^2 = A$ implies $A = 1$.
If $\text{rank}(A) = 0$, then $A = 0$.
c) If $\text{rank}(A) = 1$, then Ax is a multiple of a vector w. Assume $Au = 0$. Since by a), $Aw = w$,

and $Au = 0$, the two vectors w and u span the plane. A general vector is of the form $su + tw$. We have $Ax = tw$ so that A is a projection along u onto the line spanned by w.

Section 3.2 Basis

6) The intersection is a linear subspace, the union not.

18) Linear dependent.

24) Finding the basis of the orthogonal complement is identical to find a basis for the kernel of

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

We can take $\begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ for example.

28) The columns are linearly independent and form a basis.

48) It is the kernel of the matrix $A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. It is the image of a matrix $B = \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$.

36) The image vectors are linearly dependent too.

38)
a) See fact 3.2.6.

b) If they would not span V, we could add an other vector and still have a linear independent set.

c) Just take the vectors v_j as the columns of the matrix.

Section 3.3 Dimension.

22) $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The first and third columns are Pivot columns, so that $1, 3$

of the original are a basis of the image of A. The kernel is 3-dimensional. We can introduce

free variables s, t, u for columns $2, 4, 5$. If $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix}$, then $w = u, v = t, z = t - u, y = s, x =$

$-2s - 3t$, so that $s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a general element in the kernel.

24) Form

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where A contains the given vectors as columns. The first and third columns are pivot columns.

Therefore, the first and third vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 6 \\ 9 \\ 6 \\ 3 \end{bmatrix}$ span the subspace.

32) We look for the kernel of the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ which is already in row reduced

form. The first two columns are pivot columns. Attach free variables to the last two columns so that $w = t, z = s, x = s - t, y = -2s - 3t$, so that a general element in the kernel is

$\vec{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$

38) a) Because the image has dimensions $0, 1, 2$ or 3, the kernel has by the dimension formula dimensions $5, 4, 3$ or 2.

b) The image of T has maximal 4 dimensions (look at the matrix in reduced row echelon form, there can be maximal 4 leading 1 and therefore maximally 4 pivot columns). The possible values of the rank of T are $0, 1, 2, 3, 4$.

52) Write down a new matrix, which continas the rows of A as the columns. This is called the **transpose** of A. The first and third column of the transposed matrix A^T are Pivot columns so that the first and third row of A form a basis of the row space.

36*) No, this is not possible by the dimension formula. The dimensions of the image and kernel have to add up to 3.

56*) The hint gives the solution away already. Assume these vectors are linearly dependent, then one could have

$$c_0\vec{v} + c_1 A\vec{v} + \ldots + c_{m-1}A^{m-1}\vec{v} = \vec{0} .$$

Multiplying both sides with A^{m-1} using $A^m = 0$ shows that $c_0 = 0$. We are left with

$$c_1 A\vec{v} + \ldots + c_{m-1}A^{m-1}\vec{v} = \vec{0} .$$

Multiply both sides with A^{m-2} to see that $c_1 = 0$. etc.

Section 3.4 Coordinates

2) The vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is perpendicular to the plane so that we can take

$$S = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

as the coordinate transformation. The inverse is

$$S^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ -2 & 2 & -2 \\ 1 & 2 & 1 \end{bmatrix} /6$$

and $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.

14) $A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}$. $S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

The matrix in the coordinates of the new basis is $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

16) a) Note that the two first basis vectors are in the plane while the third is perpendicular to the plane. Therefore, in that basis, the transformation is given by the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The matrix in the standard basis is $A = SBS^{-1}$, where $S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$. We get

$$A = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix} /7 .$$

22) The vector $\vec{x}$ is $-\vec{v} + 2\vec{w}$. Flip the vector $\vec{v}$ and add two times the vector $\vec{w}$ in the picture.

26) Yes, the transformation is linear because it is given by a matrix S^{-1}.

32)* a) The hint gives the solution away see 56) above.
b) $\vec{v}_1 = A^2\vec{v}$ is mapped to $A^3\vec{v} = 0$, $\vec{v}_2 = A\vec{v}$ is mapped to $A^2\vec{v} = \vec{v}_1$ and $\vec{v}_3 = \vec{v}$ is mapped to

$A\vec{v} = \vec{v}_2$. Therefore, the matrix in that basis is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

38)* Yes, they are similar. We will learn later a general method to check such things. At this stage of the course, we have maybe to experiment a bit. For example:

1) $S_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ flips both the diagonal elements as well as the side diagonal elements.

2) $S_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ flips the side diagonals as well as the signs of the side diagonals.

3) $S_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ changes the signs of the side diagonals.

Bingo! Combining 2) and 3) achieves the goal $S = S_1 S_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ just flips the side diagonals and so

$$\begin{bmatrix} a & d \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Note that $S = S^{-1}$ in this case.

Section 4.1 Linear spaces

In problems 6-11, we have always to check three properties: 1) we can add in the set and stay in the set, we can scale an element and still are in the set and 0 is in the set.

6) Invertible matrices form not a linear space because the zero matrix is not invertible.

7) The diagonal matrices form a linear space. All three properties are easily checked to be true.

8) The upper triangular 3×3 matrices form a linear space.

9) The 3×3 matrices whos entries are ≥ 0 form no linear space. If A is such a matrix, then $-A$ is not in the space.
10) All the matrices which have a given vector v in the kernel form a linear space. If we add such matrices then $(A + B)v = Av + Bv = 0$ etc.

11) The 3×3 matrices in row reduced echelon form form no linear space. If you add two such matrices, their sum is no more in row reduced echelon form in general. For example, adding two identity matrices is no more a matrix in rref.

36) Write $A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & k \end{bmatrix}$. The matrix $AB - BA$ is $\begin{bmatrix} 0 & b & c \\ -e & 0 & 0 \\ -h & 0 & 0 \end{bmatrix}$. In order that is zero, we must have $b = c = e = h = 0$.

48) a) A general polynomial in P_4 which is even satisfies $ax^4 + bx^2 + c$. The dimension is 3.

b) A general polynomial in P_4 which is odd satisfies $ax^3 + bx$. The dimension is 2.
58) a) follows from $\cos^2(x) + \sin^2(x) = 1$.
b) we know $f'' = 0, f'(0) = 0, f(0) = 0$ which implies $f == 0$.

c) $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$ is a sum of elements in V and therefore in V.

44) Write $S = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Now $\begin{bmatrix} \vec{u} & \vec{v} & 0 \end{bmatrix}$. Since this is also $AS = $

$\begin{bmatrix} Au & Av & Aw \end{bmatrix}$, we must have $Aw = 0$, $Av = v$, $Au = u$. This means that u, v must be in the plane and w must be perpendicular to that plane. We have 2 parameters to chose a vector in the plane and 4 parameters to chose two vectors in the plane. There is additional parameter to chose a vector perpendicular to the plane. The answer is 5.

12) Is no linear subspace.

13) Is also no linear subspace.

14) This is a linear subspace.

15) Als this is a linear subspace.

Section 5.1 Projection.

6) The angle satisfies $\cos(\alpha) = \vec{u} \cdot \vec{v}/(||\vec{u}||||\vec{v}||) = -3/(\sqrt{10} \cdot 3\sqrt{6}) = -1/(2\sqrt{15})$.

10) $2 + 3k + 4 = 0$ implies $k = -2$.

16) The vectors $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ have length 1. We just have to find one vector $\vec{u}_4$ which is perpendicular to all three and has length 1. There are different ways to solve this problem:

- One possibility is to form a matrix A which has the vectors $\vec{u}_i$ as row vectors and to compute the **kernel** of A.

- A second possibility is to take any vector $\vec{v}$ and to find the **orthogonal projection** P of $\vec{v}$ onto the space spanned by $\vec{u}_1, \vec{u}_2, \vec{u}_3$. The vector $\vec{u} = \vec{v} - P(\vec{v}$ is perpendicular to the three given ones and can be normalized.

- A third possibility to solve the problem is to see that all vectors have the same $\pm 1/2$ coordinates and look at the patterns $(+ + --), (+ - +-), (+ + ++)$ of the signs. The only $\pm$ pattern which does not occur from the vectors $\vec{u}_i$ and its negative is $(+ - -+)$.

  Indeed the vector $\vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ is perpendicular to all others and has length 1.

There are two solutions to the problem. One can take $\vec{u}_4$ or its negative.

20) We want to relate the correlation coefficient with the slope of the line $y = mx + b$. The correlation coefficient is $\cos(\alpha) = \vec{x} \cdot \vec{y}/(||\vec{x}||||\vec{y}||)$. In order that $\vec{x}$ is perpendicular to $m\vec{x} - \vec{y}$ we must have $m = \vec{x} \cdot \vec{y}/||\vec{x}||^2 = \cos(\alpha)||\vec{y}||/||\vec{x}||$.

28) The three vectors $\vec{v}_i$ are orthogonal but not yet normalized. An orthonormal basis of the three dimensional subspace is $\vec{w}_i = \vec{v}_i/2$. The projection is

$$P\vec{x} = (\vec{x} \cdot \vec{w}_1)\vec{w}_1 + (\vec{x} \cdot \vec{w}_2)\vec{w}_2 + (\vec{x} \cdot \vec{w}_3)\vec{w}_3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_3 = \begin{bmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}.$$

38) In two dimensions, since $\vec{v}_1$ and $\vec{v}_2$ enclose and angle $\pi/3$, the angle has to be $\pi/3$ degrees. In three dimensions, the third vector can be in the cone of vectors which form an angle $\pi/3$ degrees with the first vector. In n-dimensions, the set of unit vectors which form an angle $\pi/3$ with the first vector form a $n - 2$ dimensional sphere. The angle between two vectors in this sphere can again be anything between $2\pi/3$ (when all three vectors are in the same plane) and 0.

14) $EA = \tan(\alpha), EB = \tan(\beta)$. The forces in the x directions match. The forces in the y direction are the same W. Now $F_1 = \sqrt{W^2 + \tan^2(\alpha)}, F_2 = \sqrt{W^2 + \tan^2(\beta)}$. Leonardos analysis is not correct for $W > 0$.

Section 5.2 Coordinates

2) Normalize $\vec{w}_1 = \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}$, then form $\vec{u}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \vec{v}_2$ and normalize $\vec{w}_2 = \vec{v}_2/7$.

14) $\vec{w}_1 = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}$.

$\vec{w}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

$\vec{w}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

16) $Q = \begin{bmatrix} 6/7 & 2/7 \\ 3/7 & -6/7 \\ 2/7 & 3/7 \end{bmatrix}$. $R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

34) To compute the kernel, we bring the matrix A into row reduced echelon form: $\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, find find a basis for the kernel and then make it orthogonal using Gram-Schmidt.
We end up with $\vec{w}_1 = [1, -2, 1, 0]^T/\sqrt{6}$ and $\vec{w}_2 = [2, -1, -4, 3]^T/\sqrt{30}$.

40) The matrix R will be diagonal.

42) The product is the area of the parallelepiped spanned by the two vectors.

Section 5.3 Projection

6) Yes, we have to show that if $A^T A = I_n$, then also $AA^T = B^T B = I_n$, where $B = A^T$.

From $A^T A = I_n$, we get $A^T = A^{-1}$ which can be read as $B = (B^T)^{-1}$. Taking inverses shows $B^{-1} = B^T$. By multiplying both sides from the right with B, we end up with $B^T B = I_n$.

(It is useful to remember from this that also the rows of an orthogonal matrix form an orthormal basis.)

8) a) No, take just the example on the same page, where $A^T A$ is I_2 while AA^T is a projection matrix.

b) Yes, it is the case because we have seen in problem 6) that $B = A^T$ is also $AA^T = I_n$.

16) Yes: $A = A^T$ implies $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$.
Note that if A, B are different symmetric matrices, then AB is not necessary symmetric any more.

20) The two vectors are not yet orthonormal, but a Gram-Schmidt orthonormalisation gives

to normal vectors which can be used to define $A = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}$ and get the projection

$$P = AA^T = \begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix}/50.$$

40) The two vectors are not yet orthonormal. We do Gram-Schmidt orthogonalization:

$$\vec{w}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}/10$$

The matrix of the orthogonal projection is QQ^T, where

$$Q = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}.$$

which is

$$\begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 12 \end{bmatrix}/50 .$$

18) a) $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

b) Actually, A^2 is symmetric, because $(A^2)^T = (AA)^T = (-A^T)(-A^T) = (A^T)^2$.

44) We can write any quaternion matrix as $M = p1 + qi + sj + rk$, where

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

From this is is obvious that M is a linear space of dimension 4. Because $i.j = -i.j = k, j.k = -k.j = i, k.i = -i.k = j, i.i = j.j = k.k = -1$, the product of quaterion matrices is a quaternion matrix. We compute $M^T M = (p^2 + q^2 + r^2 + s^2)I_4$ and see from this that M is invertible if $(p^2 + q^2 + r^2 + s^2)$ is not zero. In the other case, we have the zero matrix. The inverse is in

M too because $M^{-1} = M^T/(p^2 + q^2 + r^2 + s^2)$ and $M^T = -M + pI_4$ is a quaternion matrix too. Quaternions do not commute because $ij = -ji$.

ROTATION WITH QUATERNIONS. One can calculate with quaternions directly without the matrix representation. One often writes $q = (s, \vec{p}) = s + ip_1 + jp_2 + kp_3$. Quaternions are useful to compute rotations in space. If one wants to rotate a vector $\vec{v}$ in space by an angle ϕ around an axis which contains the unit vector $\vec{u}$, one can form the quaternions $q = (\phi, \vec{u})$ and $p = (0, \vec{v})$ and form the new quaternion $p' = qpq^{-1}$. It has the form $p' = (0, \vec{v}')$, where $\vec{v}'$ is the rotated vector. This algebraic manipulation is useful in physics or computer graphics.
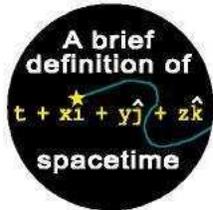
DISCOVERY. Quaternions were discovered by William Rowan Hamilton while walking along the Royal Canal. He was so excited about his "invention" that he wrote the properties $i^2 = j^2 = k^2 = ijk = -1$ of the quaternions into the stone of the Brougham bridge. Quaternions are also called **hypercomplex numbers**. They are not only used in computer graphics, also physisists find them handy.
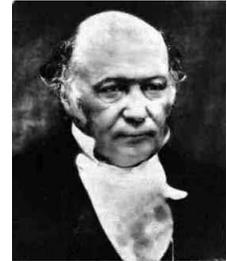
Section 5.4 Orthogonality and least squares

2) The kernel of A^T is equal to the orthogonal complement of the image of A. Since the later is a plane, the kernel of A^T is a line. To find the kernel of $B = A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, we do row reducution and get $\text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ from which we see that $[1, -2, 1]^T$ is a basis for the kernel.

10) a) Consistent means that we have a solution $A\vec{v} = \vec{b}$. The hint leads the path: split up this solution $\vec{v} = \vec{v}_h + \vec{v}_0$, where $\vec{v}_h$ is in the kernel of A and $\vec{v}_0$ in the complement. Now, $A\vec{v}_h = 0$ implies $A\vec{v}_0 = \vec{b}$ also and we found the solution.
b) If there were two solutions x_0, y_0 in the orthogonal complement of the kernel, then the element $\vec{v} = \vec{x}_0 - \vec{y}_0$ in the orthogonal complement of the kernel satisfies $A\vec{v} = \vec{0}$. But because $\vec{v}$ is both in the kernel and the orthogonal complement, it is $\vec{0}$ and $\vec{x}_0 = \vec{y}_0$.
c) This follows from Pythagoras and the relation $\vec{x}_1 = \vec{x}_0 + \vec{x}_h$.

22) Use the routine formula $(A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. The error is 0, the system is actually constistent.

34) $f(t) = 1.5 - 1.53669\cos(t) + 0.0431477\cos(2t) + 0.108974\sin(t) + 0.302692\sin(2t)$. It was of course ok to use technology here.
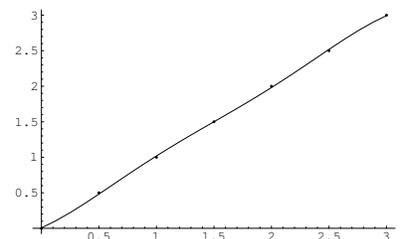
40) From $D = ka^n$ we get $\log(D) = \log(k) + n\log(a)$. We We fit the data points $(\log(a_i), \log(D_i))$ with respect to the functions $1, t$ and get the best linear fit $f(s) = 0.00048689 + 1.49982s \sim 3s/2$. This means that $g(t) = e^{3t/2}$ is the best fit.

$D = a^{3/2}$ means $D^2 = a^3$ which is Keplers third law.

The constant K is one because units have been chosen so that for the earth the constant is $K = 1$.

16) Assume A is a $m \times n$ matrix so that A^T is a $n \times m$ matrix Apply dimensions to $\mathrm{im}(A)^\perp = \ker(A^T)$ gives $m - \mathrm{rank}(A) = m - \mathrm{rank}(A^T)$, where we have used that $\dim\ker(A^T) + \dim\mathrm{ran}(A^T) = m$.

18) The rank of $A^T A$ is smaller or equal then the rank of A. Because the kernel of A^T is perpendicular to the image of A, the rank of $A^T A$ is indeed equal to the rank of A. and because the ranks of A and A^T agree also the rank of $A^T A$ is equal to the rank of AA^T.

Section 6.1 Determinants
8) The sum of the first and third column is twice the second column. The matrix is not invertible. The determinant is 0.

18) The determinant is 45-10k. The matrix is invertible for k different from 4.5.

34) The fastest way is to notice that this is a partitioned matrix. The determinant is the product of the $2x2$ matrices in the diagonal which is $9 \cdot (-5) = -45$.

40) Do Lagrange expansion repetitively to get -120. You could also do some permutations of rows to end up with a diagonal matrix.

44) $\det(kA) = k^n \det(A)$. We scale every row by a factor k and there are n rows.

Section 6.2 Determinants
6) After row reduction, end up with a matrix M_{n-1}. Because $M_1 = 1$, we have $M_n = 1$ for all n.

8) Moving the first row to the end needs 4 row swaps and produces a triangular matrix with determinant 2.

16) We have $t^2(b - a)$. Because for $a = t$ and $b = t$, the matrix is not invertible, we see that $f(t) = c(t - a)(t - b)$ and from a) we get $f(t) = (a - b)(t - a)(t - b)$. The matrix is invertible if $a \neq b$ and t is different from both a and b.

42) Because $A^T A = R^T Q^T Q R = R^T R$, the determinant of $A^T A$ is the same as the determinant of $R^T R$ which is the product of the determinants of R^T and the determiant of R which are both $r_{11} \cdots r_{nn}$. Therefore, the product has determinant $r_{11}^2 \cdots r_{nn}^2$.

Section 6.3 Determinants and geometry

6.3. 14. The volume is $\sqrt{\det(A^T A)}$ with $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, which is $\sqrt{6}$.

Section 7.1 Eigenvalues and Eigenvectors
38ab) From the description we know the images of the basis vectors and so the columns of A:
$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, a matrix, which indeed has the given eigenvectors $\vec{v}_1$ with eigenvalue 2 and $\vec{v}_2$ with eigenvalue -1.
c) Because $\vec{e}_1 = (\vec{v}_1 + \vec{v}_2)/3$, we have $A^n \vec{e}_1 = (A^n \vec{v}_1)/2 + (A^n \vec{v}_2)/3 = 2^n \vec{v}_1/3 + (-1)^n \vec{v}_2/3 = \begin{bmatrix} 2^n/3 + 2(-1)^n/3 \\ 2^n/3 - 1(-1)^n/3 \end{bmatrix}$.

50) To solve the problem we have to find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.
and find $A^n \vec{v}$ for $\vec{v} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}, \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, and $\begin{bmatrix} 600 \\ 500 \end{bmatrix}$. The eigenvalue $\lambda_1 = 3$ has the eigenvector
$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the eigenvalue $\lambda_2 = 2$ has the eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a) The vector $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$ is just an eigenvector and therefore $A^n \begin{bmatrix} 100 \\ 100 \end{bmatrix} = 2^n \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

b) The vector $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$ is also an eigenvector and $A^n \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 3^n \begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

c) The vector $\begin{bmatrix} 600 \\ 500 \end{bmatrix}$ is $4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + 1 \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ so that $A^n \begin{bmatrix} 200 \\ 100 \end{bmatrix} = 4 \begin{bmatrix} 2^n 100 \\ 2^n 100 \end{bmatrix} + \begin{bmatrix} 3^n 200 \\ 3^n 100 \end{bmatrix}$.

Section 7.2 Finding the eigenvalue

8) The characteristic polyonomial is $x^3 - 3x^2 = x^2(x-3)$ which shows that there are two eigenvectors to the eigenvalue 0 and one eigenvector to the eigenvalue 3. The eigenvector to 3 is $[1, 1, 1]^T$, the eigenvectors to 0 are spanned by $[1, -, 1, 0], [0, 1, -1]$. Geometrically, A is a projection onto the plane $x + y + z = 0$ followed by an dilation by a factor 3.

28) a) $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ which has an eigenvalue 1 with eigenvector $[1, 2]^T$. There is an other eigenvalue 0.7 with eigenvector $[-1, 1]^T$.

b) We have $[1200, 0]^T = 400[1, 2]^T + 800[1, -1]^T$ so that $[w(n), n(t)]^T = 400[1, 2]^T + 800 \cdot 0.7^n [1, -1]^T$.

c) As $n \to \infty$, the situation stabelizes at $\begin{bmatrix} 400 \\ 800 \end{bmatrix}$, so that family Wipf survives the supermaket assault.

38) a) Because $x^3 + 6x$ has a positive derivative, the function is monotone and invertible.

b) Cardano claims that if $v - u = x, uv = 2v^3 - u^3 = 20$, then x solves $x^3 + 6x = 20$. Indeed, $x^3 = (v - u)^3 = v^3 - u^3 - 3v^2 u + 3vu^2 = 20 - 3vu(u - v) = 20 - 6x$.

c) From the second equation get $u = 2/v$, then get $v^3 - 8/v^3 = 20$ which has the real solutions $v = 1 - \sqrt{3}$ and $v = 1 + \sqrt{3}$ and $u = (-1 + \sqrt{3})$ or $(-1 - \sqrt{3})$. Therefore, $x = v - u = 2$

d) This is the same computation but with constants. If p is negative, then step a) goes wrong.

e) Just plug in in $x = t - a/3$ into $x^3 + ax^2 + bx + c$ to get $t^3 + (b - a^2/3)t + (2a^3/17 - ab/3 + c)$ which is the Cardano form.

Section 7.3 : Finding the eigenvalues of a matrix

8) The matrix is upper triangular and has therefore as eigenvalues the diagonal elements of A which are $1, 2, 3$. Because the eigenvalues are different, the geometric multiplicity of each eigenvalue is 1. The eigenvalue 1 has the eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. To get the eigenvector to the eigenvalue 2, compute the kernel of $A - 2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. To get the eigenvector to the eigenvalue 3, compute the kernel of $A - 3 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ which is row reduced the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ and has the kernel $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

16) The characteristic polynomial is $\lambda(\lambda^2 + 1)$. The matrix has the eigenvalues $0, i, -i$. The eigenvector to the eigenvalue 0 is the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. It is now enough to compute the eigenvector $\vec{v}$ to the eigenvalue i because the conjugate vector $\overline{\vec{v}}$ is the eigenvector to $-i$. The eigenvector to i is $\begin{bmatrix} 1 \\ i - 1 \\ 2 \end{bmatrix}$, the eigenvector to $-i$ is $\begin{bmatrix} 1 \\ -i - 1 \\ 2 \end{bmatrix}$.

20) The eigenvalues 1 has algebraic multiplicity 2. The eigenvalue 2 has algebraic multiplicity 1. The geometric multiplicity of the eigenvalue 2 does not depend on a, b, c. Look at $\ker(A - 1) = \ker(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 2 \end{bmatrix}$. You see that this is two dimensional if $a = 0$ and one dimensional if $a \neq 0$. Therefore, for $a = 0$, the geometric multiplicity of 1 is 2 and we have an eigenbasis. For $a \neq 0$, the geometric multiplicity of 1 is 1 and we have no eigenbasis.

28) This matrix is a Jordan normal block. Since $J - k$ has rank $n - 1$, the kernel is one dimensional. The only eigenvalues of J is k. It has algebraic multiplicity n and geometric multiplicity 1.

42) From $C(t + 1) = 0.8C(t) + 10$, we get

$$A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix}.$$

This matrix has an eigenvector $\vec{v}_1 = [50, 1]$ to the eigenvalue 1 and an eigenvector $\vec{v}_2 = [1, 0]$ to the eigenvalue 0.8. The vector $\vec{v} = [0, 1]$ can be written as $\vec{v} = \vec{v}_1 - 50\vec{v}_2$ and so $A^n\vec{v} = A^n\vec{v}_1 - 50A^n\vec{v}_2 = \vec{v}_1 - 50(0.8)^n = [50(1 - 0.8^n), 1]$. Therefore $C(t) = 50(1 - 4^t/5^t)$. The function is 0 at $t = 0$, increases monotonically for increasing t and reaches asymptotically, the value 50.

48) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda_1 = (1 + \sqrt{5})/2$ with eigenvector $\vec{v}_1 = [\lambda_1, 1]$ and the eigenvalue $\lambda_2 = (1 - \sqrt{5}/2$ with eigenvector $\vec{v}_2 = \lambda_2, 1]$. Because $[1, 0] = (\vec{v}_1 - \vec{v}_2)/\sqrt{5}$, we know

$A^n \vec{v} = (A^n \vec{v}_1 - A^n \vec{v}_2)/\sqrt{5} = (\lambda_1^n \vec{v}_1 - \lambda_2^n \vec{v}_2)/\sqrt{5}$. So $j(n) = (\lambda_1^n - \lambda_2^n)/\sqrt{5}, a(n) = j(n+1)$. In the limit $n \to \infty$, the term λ_2^n vanishes and we have $j(n+1)/j(n) \to \lambda_1$, the golden ratio.

Section 7.4 : Diagonalization

14) The matrix has three different eigenvalues and is therefore diagonalizable. The eigenvectors are $[1,0,0], [-1,1,0], [0,-1,1]$. The matrix S, diagonalizing A is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

18) The matrix has the eigenvalues $0, 1, 2$, which are different, so A is diagonalizable. The eigenvectors are $[1,0,1], [0,1,0], [-1,0,1]$. The matrix S diagonalizing A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

20) To investigate the geometric multiplicities of the eigenvalue 1, look at the kernel of $A - 1 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$. If a is zero, the geometric multiplicity is 2 otherwise, the geometric multiplic-ity is 1. To investigate the geometric multiplicities of the eigenvalue 2, look at the kernel of $A - 2 = \begin{bmatrix} -1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$. The geometric multiplicity is always 1. There is an eigebasis for $a = 0$.

26) The determinant is the product of the eigenvalues. If the determinant is negative, there must be an odd number of negative eigenvalues and an odd number of positive eigenvalues. Since 0 is odd, there is at least one positive eigenvalue.

36) The first matrix has the eigenvalues $2, 3$ and both have the same trace and determinant so that both matrices have the same eigenvalues. Because both matrices are diagonalizable, they are similar.

54) If they were, also A^2 and B^2 were similar but $A^2 = 0$ and $B^2 \neq 0$. The matrices are not similar.

58) This is the Cayley-Hamilton theorem. Since it is true for the diagonalized matrix B, we apply S to the left and S^{-1} to the right of the identity $f_A(B) = 0$ to get also $f_A(A) = 0$.

56) The characteristic polynomials of both sides in the hint are the same. Because of the prod-uct property of the characteristic polynomial, the characteristic polynomial of the left hand side is the one of AB, while the characteristic polynomial of the right hand side is the one of BA.

Section 7.5 : Complex Eigenvalues

8) $(\cos(3\theta) + i\sin(3\theta) = e^{3i\theta} = (e^{i\theta})^3 = (\cos(\theta) + i\sin(\theta))^3$. So: $\boxed{\cos(3\theta)} + i\ \boxed{\sin(3\theta)}$
$= (\cos(\theta) + i\sin(\theta))^3 = \boxed{\cos^3(\theta) - \cos(\theta)\sin^2(\theta)} + i\ \boxed{(\cos^2(\theta)\sin(\theta) - \sin^3(\theta))}$.

12) The characteristic polynomial of the complex conjugate $\overline{A}$ is $\overline{f}_A(\lambda)$, the polynomial, where all coefficients are conjugated. But this is the same as the characteristic polynomial of A be-cause A is real. Since $f_A(\lambda) = \prod_i(\lambda - \lambda_i) = \overline{f}_A(\lambda) = \prod_i(\lambda - \overline{\lambda_i})$, the fundamental theorem of algebra assures that the set $\{\overline{\lambda_i}\}$ is the same as the set $\{\lambda_i\}$.

24) The characteristic polynomial is $\lambda^3 - 3\lambda^2 + 7\lambda - 5$ which has three roots $1, 1 + 2i, 1 - 2i$.

There is an eigenvector $[1, 1, 1]$ to the eigenvalue 1. The eigenvector to $1+2i$ is $[-3+4i, 5+10i]$, the eigenvector to $1-2i$ is $[-3-4i, 5-10i]$.

32) The transition matrix is

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

If you use the power method, you will see that $(0.4, 0.35, 0.25)$ is the stable equilibrium. Indeed $(0.4, 0.35, 0.25)$ is an eigenvector of A. Instead of using the power method, you could have computed the eigenvector to the eigenvalue 1. Note that these matrices always have an an eigenvalue 1, because A^T has the property that the sum of the columns is 1 and A^T therefore has the eigenvector $(1, 1, 1)$. Because A^T and A have the same eigenvalues, also A has the eigenvalue 1. The corresponding eigenvector is the stable equilibrium.

38) a) C, C^2, C^3, A^4 are again permutation matrices.

b) The matrix has the characteristic polynomial $\lambda^4 - 1$ and the eigenvalues $1, i, -1, -i$.

c) The circulant matrix M is the sum $aC^0 + bC^1 + cC^2 + dC^3$ and has the same eigenvectors than C. The eigenvalues of M are $a + b\lambda + c\lambda^2 + d\lambda^3$.

Section 7.6: Stability
8) To determine whether $(0,0)$ is a stable equilibrium of the dynamical system $x(t+1) = Ax$, where $A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$, we compute the trace and the determinant and look whether $|\text{tr}(A)| - 1 < \det(A) < 1$. In this case, where $\text{tr}(A) = 1.7, \det(A) = 0.72$, we have stability.

20) This is a rotation-dilation matrix with eigenvalues of A are $4 \pm 3i$.
1. Solution. The dynamics is a dilation by a factor $\sqrt{4^2 + 3^2} = 5$ composed by a rotation with the angle $\alpha = -\arctan(3/4)$. Applying the matrix A corresponds to applying a multiplication with the complex number $z^n = 4 - 3i$. We have $z^n = 5^n e^{in\alpha} = 5^n \cos(n\alpha) - 5^n i \sin(n\alpha)$. The initial vector corresponds to the complex number i. Therefore,

$$A^n \vec{v} = \begin{bmatrix} -5^n \sin(n\alpha) \\ 5^n \cos(n\alpha) \end{bmatrix}.$$

2. Solution. $B = S^{-1}AS = \begin{bmatrix} 4 - 3i & 0 \\ 0 & 4 + 3i \end{bmatrix}$ is diagonal, where $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ contains eigenvectors as columns. Since $B^n = S^{-1}A^n S = \begin{bmatrix} (4 - 3i)^n & 0 \\ 0 & (4 + 3i)^n \end{bmatrix}$. We get

$$A^n \vec{v} = S \begin{bmatrix} (4 - 3i)^n & 0 \\ 0 & (4 + 3i)^n \end{bmatrix} S^{-1} \vec{v} = \begin{bmatrix} \frac{1}{2i}[(4 + 3i)^n - (4 - 3i)^n)] \\ \frac{1}{2}[(4 + 3i)^n - (4 - 3i)^n)] \end{bmatrix}.$$

42) a) We can write the system as $T(x, y) = (x - ky, y + k(x - ky)) = (x - ky, kx + (1 - k)y)$. The corresponding matrix is $A = \begin{bmatrix} 1 & -k \\ k & (1 - k^2) \end{bmatrix}$.
b) All eigenvalues of the matrix A satisfy $|\lambda| = 1$ because $\det(A) = 1$, $\text{tr}(A) = 2 - k^2 < 2$.

28) If $x(t+1) = Ax$ has a stable origin, then the eigenvalues of A satisfy $|\lambda| < 1$. The matrix A has the eigenvalues $\lambda + 2$. They can no more satisfy $|\lambda| < 1$ and the origin is unstable.

38) a) $Aw + b = w$ means or $(A - 1)w = -b$ or $w = (A - 1)^{-1}b$.
The affine transformation can be modeled by a linear transformation $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. The eigenvalues of B are the eigenvalues of A and 1. If all eigenvalues of A are smaller then 1, then $T^n \vec{v}$ converges to $\vec{w}$.

Section 8.1: Symmetric matrices
10) The eigenvalues are 0 and 9. An eigenvector to 9 is $[1, -2, 2]$, eigenvectors to 0 are $[1, -2, 2]$ and $[-2, 0, 1]$. Normalize them to get an orthonormal eigenbasis.

24) The characteristic polynomial is $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2$. We can find eigenvectors $[1, 0, 0, 1], [0, 1, 1, 0]$ to the eigenvalues 1 and $[-1, 0, 0, 1], [0, -1, 1, 0]$ to the eigenvalues -1.

4) $\frac{d}{dt}P = 0.03P$ with $P(0) = 7$ has the solution $e^{0.03t}7$. This is an exponentially increasing function.

8) Seperation of variables gives $dx/\sqrt{x} = dt$. After integration, we have $2\sqrt{x} = t + C$ and so $x(t) = (t + C)^2/4$. With the initial condition $x(0) = 4$, we get $x(0) = 4 = (0 + C)^2/4$, so that $C = 16$. The solution is $x(t) = (t + 4)^2/4$. Note that there is also a solution $x(t) = (t - 4)^2/4$. The differential equation has no unique solution. This is possible, because $x''(0)$ is not defined. The function $f(x) = \sqrt{x}$ is not smooth at $x = 0$. It has an infinite derivative there.

10) Separate the variables $x' = 1/\cos(x)$ is equivalent to $dx\cos(x) = dt$ or $\sin(x) = t + c$ so that $x = \arcsin(t + c)$.

26) The matrix A of the equation $\dot{x} = Ax$ has the eigenvalues $-2, 3$ with eigenvectors $[-2, 3]^T, [1, 1]^T$. The initial condition $[7, 2]^T$ can be written as $(-1)[-2, 3]^T + 5[1, 1]^T$ so that The solution is $x(t) = 2e^{-2t} + 5e^{3t}, y(t) = -3e^{-2t} + 5e^{3t}$.

32) (This is the same matrix as in 26). The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ has the eigenvalues $3, -2$ with eigenvectors $[-2, 3]^T, [1, 1]^T$. The lines which contain these vectors form the asymptotic lines of a family of hyperbola, the orbits of the system.

Section 9.1:

54) a) With the matrix $A = \begin{bmatrix} 0 & 1 \end{bmatrix}$, the system can be written as $\dot{x} = Ax$. The matrix A has the eigenvalues $-2, -1$ with eigenvectors $[1, -1]^T$, $[1, -2]^T$. The system is asymptotically stable. All orbits are attracted by the equilibrium point $(0,0)$. There are special orbits which are located on the lines containing the eigenvectors.

b) The two eigendirections partition the phase space into four regions. If one starts in any of these regions, one stays in any of these regions. The lower left region (see picture) consists of orbits, where the door slams.
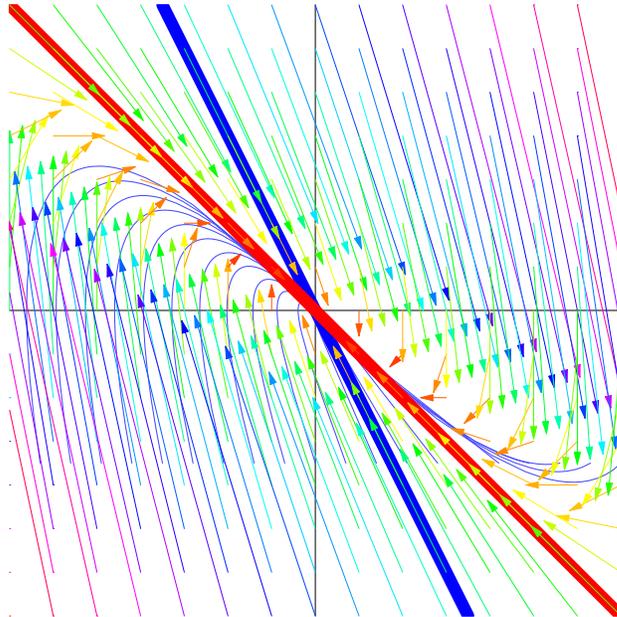
Section 9.2:

12) Determine the stability of $\dot{x} = Ax$ with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}$. The system has an eigenvalue < 1 and 2 complex conjugate eigenvalues. The system is Liouville stable (orbits which start near the origin stay near the origin), but not asymptotically stable.

14) a) The constants k_i provide decay rates for each mode. The entry b provides a feedback from the last mode to the first with a negative effect: increasing x_n decreases x_1.

b) The matrix is $A = \begin{bmatrix} -k_1 & -b \\ 1 & -k_2 \end{bmatrix}$ which has positie determinant $k_1 k_2 + b$ and negative trace $k_1 + k_2$. The origin is asymptotically stable.

c) There is no stability in general. The matrix is for $k_1 = k_2 = k_3 = 1$ equal to $A = \begin{bmatrix} -1 & 0 & b \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. While for small b all eigenvalues are negative, for large b, there is a positive eigenvalue because the determinant becomes positive and three negative eigenvalues would not be possible.

18) If the system is stable, then the all three eigenvalues are negative. The determinant has to be negative and the trace has to be negative too.

22) pairs with VII
23) pairs with II
24) pairs with I
25) pairs with IV
26) pairs with V

34) The matrix $A = \begin{bmatrix} 7 & 10 \\ -4 & -5 \end{bmatrix}$ has the eigenvalues $1 \pm i2$ with eigenvectors $[-3/2 - i/2, 1]^T$ and $[-3/2 + i/2, 1]^T$. Both $x(t)$ and $y(t)$ are of the form $ae^t \cos 2t + be^t \sin(2t)$. Solving gives $x(t) = e^t(\cos(2t) + 3\sin(2t))$ and $y(t) = e^t(-2\sin(2t))$.

Section 8.4 Nonlinear systems
(see seperate solution page)
Section 4.2 Linear Transformations
28) $T(f) = f(2t) - f(t)$ is linear. For $f(x) = ax^2 + bx + c$, the image is $T(f)(x) = a(2x)^2 + b(2x) + c - (ax^2 + bx + c) = 3ax^2 + 2bx$. The image is a two dimensional space and does not contain constants. The map is not an isomorphism.

40) The map $T(f) = f'' + 2f' + f$ is a differential operator, linear as well as an isomorphism on C^∞.

34) The shift map $T(x_0, x_1, x_2, ...) = (0, x_1, x_2, ...)$ is a linear map but not an isomorphism. The element $(1, 0, 0, ...)$ for example is not in the image.

58) The image of $T(x_0, x_1, x_2, ...) = (0, x_1, x_2, ...)$ are all sequences for which $x_0 = 0$. The kernel is trivial.

66) The map $T(f) = f - f'$ has the one dimensional kernel spanned by $f(x) = e^x$. The nullety is 1.

Section 9.3 Nonlinear systems
6) $f'(t) + 2f(t) = e^{3t}$.
1. Solution.
The homogenous equation has the solutions $f(t) = Ce^{-2t}$. A special solution is obtained by plugging Ae^{3t} into the equation which gives $3Ae^{3t} + 2Ae^{3t} = e^{3t}$ so that $A = 1/5$. The solution is $Ce^{-2t} + e^{3t}/5$.

2. Solution. We use the operator method. The equation can be written as $(D+2)f = e^{et} = g$ and therefore, $f = (D + 2)^{-1}g = e^{-2t} \int_0^t e^{2t}e^{3t}\ dt = e^{-2t}(e^{5t}/5 + C)$.

28) $f''(t) + f'(t) - 12f(t) = 0$, $f(0) = f'(0) = 0$.
The equation is $(D^2 + D - 12)f = (D - 3)(D + 4)f = 0$ which has the solutions $C_1e^{3t} + C_2e^{-4t}$. The initial conditions tell us that $C_1 + C_2 = 0, 3C_1 - 4C_2 = 0$ leading to $C_1 = C_2 = 0$. The function $f(t) = 0$ is the unique solution.

34) The downward gravitational force $F_1 = gm$, where $m = \rho L^3$ is the mass of the block and g is the gravitational constant and L is the length. So $F_1 = g\rho L^3$.

The buoyancy is $F_2 = -gxL^2$. The total force is $F = F_1 + F_2 = gL^3 - gxL^2$.

b) Newton's law gives $mx''(t) = F(x(t))$. With $m = \rho L^3$ we obtain $\rho L^3 x''(t) = L^2g(L - x(t))$ or $x''(t) + ax(t) = b$, where $a = g/(L\rho)$ and $b = g$.

c) We have a constant solution $x(t) = b/a = gL\rho/g = L\rho = 8$. The solution of the homogeneous equation $x''(t) + ax(t) = 0$ is $x(t) = c_1\cos(\sqrt{a}t) + c_2\sin(\sqrt{a}t)$. With the initial condition $x(0) = 10$ and $x'(0) = 0$, we obtain $x(t) = 8 + 2\cos(\sqrt{a}t)$.

d) The frequency increases if g increases. It decreases if L or ρ increases. A heavier wood will bounce faster. The wood would bounce slower on mars.

42) a) Every real λ is an eigenvalue. For positive λ we can take $f(t) = e^{\sqrt{\lambda}t}$, for $\lambda = 0$ we can take $f(t) = 1$, for negative λ, we can take $\cos(\sqrt{\lambda}t)$.

b) For positive λ, the function is not periodic. For nonpositive λ, $\lambda = -n^2\pi^2$, we have the eigenfunction $f(t) = \cos(n\pi t)$. The eigenvalues are $-n^2\pi^2$, where $n = 0, 1, 2, ...$

44) The differential equation is $f'' + 4f + 5f = \cos(3t)$. We can write this as $(D - (-2 - i))(D + (-2 + i))f = \cos(3t)$. The homogeneous problem is a damped oscillator $f(t) = C_1 e^{-2t}\cos(t) + C_2 e^{-2t}\sin(t)$. A special solution can be obtained with the "Ansatz" $f_h(t) = A\cos(3t) + B\sin(3t)$ because the inverse operator $(D - \lambda)^{-1}$ leaves this space invariant. Plugging this into the equation and comparing coefficients gives $A = -3/40, B = 1/40$ so that $f_p = -\cos(3t)/40 + 3\sin(3t)/40$ is a particular solution. The general solution is $f(t) = C_1 e^{-2t}\cos(t) + C_2 e^{-2t}\sin(t) + -\cos(3t)/40 + 3\sin(3t)/40$.

b) For $t \to \infty$, the homogeneous solution will die out and the motion will approach the particular solution.

Homework for Section 5.5 **Math 21b, Fall 2004**

Recall: In this homework, we look at the **inner product space** with

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx \ .$$

One can define length, distances or angles in the same way as we have done with the dot product for vectors in R^n. Functions are assumed to be (piecewise) smooth.

Homework for first lesson (inner product spaces)

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos \frac{\langle f,g \rangle}{\|f\|\|g\|}$ where $\|f\| = \sqrt{\langle f, f \rangle}$.)

    Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

    Solution:
    From $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(x) \, dx = \frac{1}{\pi} [2x \cos(x) + (x^2 - 2)\sin(x)] \mid_{-\pi}^{\pi} = -4$ and $\|f\|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) \, dx = 1$, $\|g\|^2 = \langle g, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos(x) \, dx = 2\pi^4/5$, we get $\alpha = \arccos(-10/\pi^4)$.

    ---

2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

    a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) \, dx$.

    b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g \rangle = \frac{2}{\pi} \int_0^{\pi} f(x)g(x) \, dx$.

    c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g \rangle = 0$.

Solution:
a) If $f(x) = f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} f(y)g(y)\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

b) If $f(x) = -f(-x)$ and $g(x) = -g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)(-g(y))\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

c) If $f(x) = -f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)g(y)\ (-dy) = -\int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 0\ .$$

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

Solution:
The square of the distance between f and g is $||f - g||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin(x) - x^2)^2\ dx = ||f||^2 + ||g||^2 - 2\langle f, g\rangle$. Since f is even and g is odd, this is $||f||^2 + ||g||^2$ (Pythagoras). The square of the distance between f and h is $||f - h||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(x) - x^2)^2\ dx = ||f||^2 + ||h||^2 - 2\langle f, h\rangle$. Because $2\langle f, h\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)x^2\ dx < 0$, the distance from f and h is smaller then the distance from g to h.

4. Determine the projection of the function $f(x) = x^2$ onto the "plane" spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h\rangle$.

Solution:
From the first problem, we know $\langle f, g\rangle = -4$. The dot product $\langle f, h\rangle$ is zero because f is even and h is odd. Because g and h are perpendicular and have length 1, the projection is $P(f) = \langle f, g\rangle g + \langle f, h\rangle h = -4g = -4\cos(x)$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.

Solution:
$||f||^2 = \langle a\cos(x) + b\sin(x), a\cos(x) + b\sin(x)\rangle = \langle a\cos(x), a\cos(x)\rangle + \langle a\cos(x), b\sin(x)\rangle + \langle b\sin(x), a\cos(x)\rangle + \langle b\sin(x), b\sin(x)\rangle = a^2\langle\cos(x), \cos(x)\rangle + b^2\langle\sin(x), \sin(x)\rangle = a^2 + b^2$.

Homework for second lesson (Fourier series)

1. Find the Fourier series of the function $f(x) = |x|$.

Solution:
The function $f(x)$ is even so that f has a cos series. From the homework on inner product, we know

$$a_0 = \frac{2}{\pi} \int_0^\pi x/\sqrt{2} \, dx = \pi/\sqrt{2} \ .$$

The other coefficients are

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx \\
&= \frac{2}{\pi} \frac{(\cos(nx)/n^2 + x\sin(nx))}{n} \Big|_0^\pi \\
&= \frac{2}{\pi} \frac{\cos(n\pi)}{n^2} \\
&= \frac{2(-1)^n}{\pi n^2} \ .
\end{aligned}
$$

The Fourier series is

$$f(x) = (\pi/\sqrt{2})\frac{1}{\sqrt{2}} + \sum_{n=1}^\infty \frac{2(-1)^n}{\pi n^2} \cos(nx) \ .$$

2. Find the Fourier series of the function $\cos^2(x) + 5\sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.

Solution:
$f(x) = 1/2 + \cos(2x)/2 + 5\sin(x) + 5 = 11\sqrt{2}/2(1/\sqrt{2}) + (1/2)\cos(2x) + 1\sin(x)$ is already the Fourier series. We have $a_0 = 11\sqrt{2}/2, a_2 = 1/2, b_1 = 1$. All other Fourier coefficients are zero.

3. Find the Fourier series of the function $f(x) = |\sin(x)|$.

Solution:

Again, the function is even.

$$a_0 = \frac{2}{\pi} \int_0^\pi \sin(x)/\sqrt{2} \, dx = \frac{2}{\pi} 2/\sqrt{2} = \frac{4}{\pi\sqrt{2}} \ .$$

The other coefficients are

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi \sin(x) \cos(nx) \, dx \\
&= \frac{2}{\pi} \frac{(\cos(x)\cos(nx) + n\sin(x)\sin(nx))}{(n^2 - 1)} \Big|_0^\pi \\
&= \frac{2}{\pi} \frac{\cos(x)\cos(nx)}{(n^2 - 1)} \Big|_0^\pi \\
&= \frac{2}{\pi} \frac{(1 + \cos(n\pi))}{(1 - n^2)} \ ,
\end{aligned}
$$

which is 0 for odd n and $2/(1 - n^2)$ for even n. The Fourier series is

$$f(x) = \frac{4}{\pi\sqrt{2}} \frac{1}{\sqrt{2}} + \sum_{n=1}^\infty \frac{2}{\pi} \frac{(1 + \cos(n\pi))}{(1 - n^2)} \cos(nx) \ .$$

The Fourier series is

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + \cdots \right)$$

4. In problem 3) you should have gotten a series

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + \cdots \right)$$

Use Parseval's identity (Fact 5.5.6 in the book) to find the value of

$$\frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \cdots$$

Solution:

Parsevals identity tells that the sum $a_0^2 + \sum_n a_n^2 = ||f||^2$. We have $||f||^2 = \frac{2}{\pi} \int_0^\pi \sin^2(x) \, dx = 1$, $a_0 = 2\sqrt{2}/\pi$ and $a_n = (-4/\pi)1/(n^2 - 1)$. Therefore,

$$\frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \cdots = (1 - 8/\pi^2)\pi^2/16 = \frac{\pi^2 - 8}{16} \ .$$

This is

$$1/3^2 + 1/8^2 + 1/15^2 + \ldots = \frac{\pi^2 - 2}{16} = 0.36685\ldots$$

Homework for third lesson (Partial differential equations)

1. Solve the heat equation $f_t = \mu f_{xx}$ on $[0, \pi]$ for the initial condition $f(x, 0) = |\sin(3x)|$.

Solution:
The Fourier series of $|\sin(3x)|$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

with

$$
\begin{aligned}
b_n \;=\; & \frac{2}{\pi} \int_0^{\pi/3} \sin(3x) \sin(nx) \, dx \\
& - \frac{2}{\pi} \int_{\pi/3}^{2\pi/3} \sin(3x) \sin(nx) \, dx \\
& + \frac{2}{\pi} \int_{2\pi/3}^{3\pi/3} \sin(3x) \sin(nx) \, dx \; .
\end{aligned}
$$

We had to break up the integral because $|\sin(3x)| = -\sin(3x)$ on the interval $[\pi/3, 2\pi/3]$. To evaluate the first integral, use the formula $2\sin(3x)\sin(nx) = \cos((3-n)x) - \cos((3+n)x)$ to get

$$b_n = \frac{1}{\pi}\left[\frac{\sin((3-n)x)}{3-n} - \frac{\sin((3+n)x)}{3+n}\right]\Big|_0^{\pi/3} = \frac{1}{\pi}\left(\frac{\sin((3-n)\pi/3)}{3-n} - \frac{\sin((3+n)\pi/3)}{3+n}\right) \; .$$

This can be simplified to $\frac{1}{\pi}\left(\frac{\sin(n\pi/3)}{3+n} + \frac{\sin(n\pi/3)}{3-n}\right) = \frac{6}{\pi}\frac{\sin(n\pi/3)}{n^2-9}$. Analoguously, the second integral is $\frac{6}{\pi}\frac{(\sin(n\pi/3)+\sin(2n\pi/3))}{n^2-9}$. The third integral is $\frac{12}{\pi}\frac{\sin(n\pi/3)}{n^2-9}$ again. Together,

$$b_n = \frac{12}{\pi}\frac{\sin(n\pi/3)}{9-n^2} + \frac{12}{\pi}\frac{\sin(n2\pi/3)}{9-n^2}$$

Note that for $n = 3$, the number exists and is $1/3$. The solution to the heat equation is

$$f(x, t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$$

---

2. We want to see in this exercise how to deal with solutions to the heat equation, where the boundary values are not 0.

   a) Verify that for any constants a, b the function $h(x, t) = (b - a)x/\pi + a$ is a solution to the heat equation.

   b) Assume we have the problem to describe solutions $f(x, t)$ to the heat equations, where $f(0, t) = a$ and $f(\pi, t) = b$. Show that $f(x, t) - h(x, t)$ is a solution of the heat equation with boundary conditions 0 at $x = 0$ and $x = \pi$.

   c) Solve the heat equation with the initial condition $f(x, 0) = f(x) = \sin(3x) + x/\pi$ and satisfying $f(0, t) = 0, f(\pi, t) = 1$ for all times t. This is a situation, when the stick

is kept at constant but different temperatures on both ends.

Solution:
a) is a simple differentiation exercise. Indeed: $h_{xx} = 0$ as well as h_t.
b) $f(x,t) - h(x,t)$ has the boundary conditions 0 at both ends.
c) The function $g(x,t) = f(x,t) - x/\pi$ satisfies $g(0,t) = 0, g(\pi,t) = 0$ and has the initial condition $g(x,0) = \sin(3x)$. We have

$$g(x,t) = e^{-9\mu t}\sin(3x)$$

and therefore

$$f(x,t) = e^{-9\mu t}\sin(3x) + x/\pi \ .$$

---

3. A piano string is fixed at the ends $x = 0$ and $x = \pi$ and initially undisturbed. The piano hammer induces an initial velocity $f_t(x,t) = g(x)$ onto the string, where $g(x) = \sin(2x)$ on the interval $[0, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$. Find the motion of the string.

Solution:
We first have to develop $g(x)$ into a Fourier sin-series:

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi/2} \sin(2x)\sin(nx)\ dx \\
&= \frac{2}{\pi}\int_0^{\pi/2} (\cos((n-2)x) - \cos((n+2)x))\ dx \\
&= \frac{2}{\pi}\Big(\frac{\sin((n-2)x)}{n-2} - \frac{\sin((n+2)x)}{n+2}\Big)\Big|_0^{\pi/2} \ .
\end{aligned}
$$

This can be simplified to

$$\frac{4}{\pi}\frac{\sin(n\pi/2)}{n^2 - 4} \ .$$

For $n = 2$, we have $b_n = 1/2$. For all other even n, we have $b_n = 0$.
The solution of the wave equation $f_{tt} = c^2 f_{xx}$ is

$$f(x,t) = \sum_n b_n \frac{\sin(nct)}{nc}\sin(nx)$$