Homework for Section 5.5 **Math 21b, Fall 2004**

Recall: In this homework, we look at the **inner product space** with

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx \,.$$

One can define length, distances or angles in the same way as we have done with the dot product for vectors in R^n. Functions are assumed to be (piecewise) smooth.

Homework for first lesson (inner product spaces)

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos \frac{\langle f,g \rangle}{\|f\|\|g\|}$ where $\|f\| = \sqrt{\langle f, f \rangle}$.)

   Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

   Solution:
   From $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(x)\ dx = \frac{1}{\pi}\left[2x\cos(x) + (x^2-2)\sin(x)\right]|_{-\pi}^{\pi} = -4$ and $||f||^2 = \langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x)\ dx = 1$, $||g||^2 = \langle g, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\cos(x)\ dx = 2\pi^4/5$, we get $\alpha = \arccos(-10/\pi^4)$.

2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

   a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)\ dx$.

   b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)\ dx$.

   c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g\rangle = 0$.

Solution:
a) If $f(x) = f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} f(y)g(y)\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

b) If $f(x) = -f(-x)$ and $g(x) = -g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)(-g(y))\ (-dy) = \int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 2\int_{0}^{\pi} f(x)g(x)\ dx$$

c) If $f(x) = -f(-x)$ and $g(x) = g(-x)$, then substitution $y = -x, dy = -dx$ gives

$$\int_{-\pi}^{0} f(x)g(x)\ dx = \int_{\pi}^{0} -f(y)g(y)\ (-dy) = -\int_{0}^{\pi} f(y)g(y)\ dy$$

so that

$$\int_{-\pi}^{\pi} f(x)g(x)\ dx = 0\ .$$

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

Solution:
The square of the distance between f and g is $||f-g||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin(x)-x^2)^2\ dx = ||f||^2 + ||g||^2 - 2\langle f, g\rangle$. Since f is even and g is odd, this is $||f||^2 + ||g||^2$ (Pythagoras). The square of the distance between f and h is $||f-h||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos(x)-x^2)^2\ dx = ||f||^2 + ||h||^2 - 2\langle f, h\rangle$. Because $2\langle f, h\rangle = \frac{1}{\pi}\int_{\pi}^{\pi}\cos(x)x^2\ dx < 0$, the distance from f and h is smaller then the distance from g to h.

4. Determine the projection of the function $f(x) = x^2$ onto the "plane" spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h\rangle$.

Solution:
From the first problem, we know $\langle f, g\rangle = -4$. The dot product $\langle f, h\rangle$ is zero because f is even and h is odd. Because g and h are perpendicular and have length 1, the projection is $P(f) = \langle f, g\rangle g + \langle f, h\rangle h = -4g = -4\cos(x)$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.

Solution:
$||f||^2 = \langle a\cos(x)+b\sin(x), a\cos(x)+b\sin(x)\rangle = \langle a\cos(x), a\cos(x)\rangle + \langle a\cos(x), b\sin(x)\rangle + \langle b\sin(x), a\cos(x)\rangle + \langle b\sin(x), b\sin(x)\rangle = a^2\langle \cos(x), \cos(x)\rangle + b^2\langle \sin(x), \sin(x)\rangle = a^2 + b^2$.

Homework for second lesson (Fourier series)

1. Find the Fourier series of the function $f(x) = |x|$.

Solution:
The function $f(x)$ is even so that f has a cos series. From the homework on inner product, we know

$$a_0 = \frac{2}{\pi}\int_0^\pi x/\sqrt{2}\, dx = 2\pi^2/21/\sqrt{2} = \pi/\sqrt{2}\,.$$

The other coefficients are

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos(nx)\, dx \\ &= \frac{2}{\pi}\frac{(\cos(nx)/n^2 + x\sin(nx)}{n})|_0^\pi \\ &= \frac{2}{\pi}\frac{\cos(n\pi)-1}{n^2} \\ &= \frac{2((-1)^n - 1)}{\pi n^2}\,. \end{aligned}$$

The Fourier series is

$$f(x) = (\pi/\sqrt{2})\frac{1}{\sqrt{2}} + \sum_{n=1}^\infty \frac{2((-1)^n-1)}{\pi n^2}\cos(nx)\,.$$

2. Find the Fourier series of the function $\cos^2(x) + 5\sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.

Solution:
$f(x) = 1/2 + \cos(2x)/2 + 5\sin(x) + 5 = 11\sqrt{2}/2(1/\sqrt{2}) + (1/2)\cos(2x) + 1\sin(x)$ is already the Fourier series. We have $a_0 = 11\sqrt{2}/2, a_2 = 1/2, b_1 = 1$. All other Fourier coefficients are zero.

3. Find the Fourier series of the function $f(x) = |\sin(x)|$.

Solution:
Again, the function is even.

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin(x)/\sqrt{2}\;dx = \frac{2}{\pi}2/\sqrt{2} = \frac{4}{\pi\sqrt{2}} \; .$$

The other coefficients are

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)\;dx \\ &= \frac{2}{\pi}\frac{(\cos(x)\cos(nx)+n\sin(x)\sin(nx))}{(n^2-1)}|_0^\pi \\ &= \frac{2}{\pi}\frac{\cos(x)\cos(nx)}{(n^2-1)}|_0^\pi \\ &= \frac{2}{\pi}\frac{(1+\cos(n\pi))}{(1-n^2)} \; , \end{aligned}$$

which is 0 for odd n and $2/(1-n^2)$ for even n. The Fourier series is

$$f(x) = \frac{4}{\pi\sqrt{2}}\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{(1+\cos(n\pi))}{(1-n^2)}\cos(nx) \; .$$

The Fourier series is

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + ...\right)$$

4. In problem 3) you should have gotten a series

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + ...\right)$$

Use Parseval's identity (Fact 5.5.6 in the book) to find the value of

$$\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots$$

Solution:
Parsevals identity tells that the sum $a_0^2 + \sum_n a_n^2 = ||f||^2$. We have $||f||^2 = \frac{2}{\pi}\int_0^\pi \sin^2(x)\;dx = 1$, $a_0 = 2\sqrt{2}/\pi$ and $a_n = (-4/\pi)1/(n^2-1)$. Therefore,

$$\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots = (1-8/\pi^2)\pi^2/16 = \frac{\pi^2-8}{16} \; .$$

This is

$$1/3^2 + 1/8^2 + 1/15^2 + ... = \frac{\pi^2-2}{16} = 0.36685....$$

Homework for third lesson (Partial differential equations)

1. Solve the heat equation $f_t = \mu f_{xx}$ on $[0, \pi]$ for the initial condition $f(x, 0) = |\sin(3x)|$.

Solution:
The Fourier series of $|\sin(3x)|$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/3} \sin(3x)\sin(nx)\,dx \\ &- \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} \sin(3x)\sin(nx)\,dx \\ &+ \frac{2}{\pi}\int_{2\pi/3}^{3\pi/3} \sin(3x)\sin(nx)\,dx\ . \end{aligned}$$

We had to break up the integral because $|\sin(3x)| = -\sin(3x)$ on the interval $[\pi/3, 2\pi/3]$. To evaluate the first integral, use the formula $2\sin(3x)\sin(nx) = \cos((3-n)x) - \cos((3+n)x)$ to get

$$b_n = \frac{1}{\pi}\left[\frac{\sin((3-n)x)}{3-n} - \frac{\sin((3+n)x)}{3+n}\right]|_0^{\pi/3} = \frac{1}{\pi}(\frac{\sin((3-n)\pi/3)}{3-n} - \frac{\sin((3+n)\pi/3)}{3+n})\ .$$

This can be simplified to $\frac{1}{\pi}(\frac{\sin(n\pi/3)}{3+n} + \frac{\sin(n\pi/3)}{3-n}) = \frac{6}{\pi}\frac{\sin(n\pi/3)}{n^2-9}$. Analoguously, the second integral is $\frac{6}{\pi}\frac{(\sin(n\pi/3)+\sin(2n\pi/3))}{n^2-9}$. The third integral is $\frac{12}{\pi}\frac{\sin(n\pi/3)}{n^2-9}$ again. Together,

$$b_n = \frac{12}{\pi}\frac{\sin(n\pi/3)}{9-n^2} + \frac{12}{\pi}\frac{\sin(n2\pi/3)}{9-n^2}$$

Note that for $n = 3$, the number exists and is $1/3$. The solution to the heat equation is

$$f(x, t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t}\sin(nx)$$

2. We want to see in this exercise how to deal with solutions to the heat equation, where the boundary values are not 0.

a) Verify that for any constants a, b the function $h(x, t) = (b - a)x/\pi + a$ is a solution to the heat equation.

b) Assume we have the problem to describe solutions $f(x, t)$ to the heat equations, where $f(0, t) = a$ and $f(\pi, t) = b$. Show that $f(x, t) - h(x, t)$ is a solution of the heat equation with boundary conditions 0 at $x = 0$ and $x = \pi$.

c) Solve the heat equation with the initial condition $f(x, 0) = f(x) = \sin(3x) + x/\pi$ and satisfying $f(0, t) = 0, f(\pi, t) = 1$ for all times t. This is a situation, when the stick is kept at constant but different temperatures on both ends.

Solution:
a) is a simple differentiation exercise. Indeed: $h_{xx} = 0$ as well as h_t.
b) $f(x,t) - h(x,t)$ has the boundary conditions 0 at both ends.
c) The function $g(x,t) = f(x,t) - x/\pi$ satisfies $g(0,t) = 0, g(\pi,t) = 0$ and has the initial condition $g(x,0) = \sin(3x)$. We have

$$g(x,t) = e^{-9\mu t}\sin(3x)$$

and therefore

$$f(x,t) = e^{-9\mu t}\sin(3x) + x/\pi \ .$$

3. A piano string is fixed at the ends $x = 0$ and $x = \pi$ and initially undisturbed. The piano hammer induces an initial velocity $f_t(x,t) = g(x)$ onto the string, where $g(x) = \sin(2x)$ on the interval $[0, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$. Find the motion of the string.

Solution:
We first have to develop $g(x)$ into a Fourier sin-series:

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} \sin(2x)\sin(nx)\ dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} (\cos((n-2)x) - \cos((n+2)x))\ dx \\ &= \frac{2}{\pi}(\frac{\sin((n-2)x)}{n-2} - \frac{\sin((n+2)x)}{n+2})_0^{\pi/2} \ . \end{aligned}$$

This can be simplified to

$$\frac{4}{\pi}\frac{\sin(n\pi/2)}{n^2-4} \ .$$

For $n = 2$, we have $b_n = 1/2$. For all other even n, we have $b_n = 0$.
The solution of the wave equation $f_{tt} = c^2 f_{xx}$ is

$$f(x,t) = \sum_n b_n \frac{\sin(nct)}{nc}\sin(nx)$$