Section 8.4 Nonlinear systems
(see seperate solution page)
Section 4.2 Linear Transformations
28) $T(f) = f(2t) - f(t)$ is linear. For $f(x) = ax^2 + bx + c$, the image is $T(f)(x) = a(2x)^2 + b(2x) + c - (ax^2 + bx + c) = 3ax^2 + 2bx$. The image is a two dimensional space and does not contain constants. The map is not an isomorphism.

40) The map $T(f) = f'' + 2f' + f$ is a differential operator, linear as well as an isomorphism on C^∞.

34) The shift map $T(x_0, x_1, x_2, ...) = (0, x_1, x_2, ...)$ is a linear map but not an isomorphism. The element $(1, 0, 0, ...)$ for example is not in the image.

58) The image of $T(x_0, x_1, x_2, ...) = (0, x_1, x_2, ...)$ are all sequences for which $x_0 = 0$. The kernel is trivial.

66) The map $T(f) = f - f'$ has the one dimensional kernel spanned by $f(x) = e^x$. The nullety is 1.

Section 9.3 Nonlinear systems
6) $f'(t) + 2f(t) = e^{3t}$.
1. Solution.
The homogenous equation has the solutions $f(t) = Ce^{-2t}$. A special solution is obtained by plugging Ae^{3t} into the equation which gives $3Ae^{3t} + 2Ae^{3t} = e^{3t}$ so that $A = 1/5$. The solution is $Ce^{-2t} + e^{3t}/5$.

2. Solution. We use the operator method. The equation can be written as $(D+2)f = e^{et} = g$ and therefore, $f = (D+2)^{-1}g = e^{-2t} \int_0^t e^{2t}e^{3t}\, dt = e^{-2t}(e^{5t}/5 + C)$.

28) $f''(t) + f'(t) - 12f(t) = 0, f(0) = f'(0) = 0$.
The equation is $(D^2 + D - 12)f = (D-3)(D+4)f = 0$ which has the solutions $C_1e^{3t} + C_2e^{-4t}$. The initial conditions tell us that $C_1 + C_2 = 0, 3C_1 - 4C_2 = 0$ leading to $C_1 = C_2 = 0$. The function $f(t) = 0$ is the unique solution.

34) The downward gravitational force $F_1 = gm$, where $m = \rho L^3$ is the mass of the block and g is the gravitational constant and L is the length. So $F_1 = g\rho L^3$.

The buoyancy is $F_2 = -gxL^2$. The total force is $F = F_1 + F_2 = gL^3 - gxL^2$.

b) Newton's law gives $mx''(t) = F(x(t))$. With $m = \rho L^3$ we obtain $\rho L^3 x''(t) = L^2 g(L - x(t))$ or $x''(t) + ax(t) = b$, where $a = g/(L\rho)$ and $b = g$.

c) We have a constant solution $x(t) = b/a = gL\rho/g = L\rho = 8$. The solution of the homogeneous equation $x''(t) + ax(t) = 0$ is $x(t) = c_1 \cos(\sqrt{a}t) + c_2 \sin(\sqrt{a}t)$. With the initial condition $x(0) = 10$ and $x'(0) = 0$, we obtain $x(t) = 8 + 2\cos(\sqrt{a}t)$.

d) The frequency increases if g increases. It decreases if L or ρ increases. A heavier wood will bounce faster. The wood would bounce slower on mars.

42) a) Every real λ is an eigenvalue. For positive λ we can take $f(t) = e^{\sqrt{\lambda}t}$, for $\lambda = 0$ we can take $f(t) = 1$, for negative λ, we can take $\cos(\sqrt{\lambda}t)$.

b) For positive λ, the function is not periodic. For nonpositive λ, $\lambda = -n^2\pi^2$, we have the eigenfunction $f(t) = \cos(n\pi t)$. The eigenvalues are $-n^2\pi^2$, where $n = 0, 1, 2, ...$

44) The differential equation is $f'' + 4f + 5f = \cos(3t)$. We can write this as $(D - (-2 - i))(D + (-2 + i))f = \cos(3t)$. The homogeneous problem is a damped oscillator $f(t) = C_1 e^{-2t} \cos(t) + C_2 e^{-2t} \sin(t)$. A special solution can be obtained with the "Ansatz" $f_h(t) = A\cos(3t) + B\sin(3t)$ because the inverse operator $(D - \lambda)^{-1}$ leaves this space invariant. Plugging this into the equation and comparing coefficients gives $A = -3/40, B = 1/40$ so that $f_p = -\cos(3t)/40 + 3\sin(3t)/40$ is a particular solution. The general solution is $f(t) = C_1 e^{-2t} \cos(t) + C_2 e^{-2t} \sin(t) + -\cos(3t)/40 + 3\sin(3t)/40$.
b) For $t \to \infty$, the homogeneous solution will die out and the motion will approach the particular solution.