Section 7.6: Stability

8) To determine whether $(0,0)$ is a stable equilibrium of the dynamical system $x(t+1) = Ax$, where $A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$, we compute the trace and the determinant and look whether $|\text{tr}(A)| - 1 < \det(A) < 1$. In this case, where $\text{tr}(A) = 1.7, \det(A) = 0.72$, we have stability.

20) This is a rotation-dilation matrix with eigenvalues of A are $4 \pm 3i$.
1. Solution. The dynamics is a dilation by a factor $\sqrt{4^2+3^2} = 5$ composed by a rotation with the angle $\alpha = -\arctan(3/4)$. Applying the matrix A corresponds to applying a multiplication with the complex number $z^n = 4 - 3i$. We have $z^n = 5^n e^{in\alpha} = 5^n \cos(n\alpha) - 5^n i \sin(n\alpha)$. The initial vector corresponds to the complex number i. Therefore,

$$A^n\vec{v} = \begin{bmatrix} -5^n \sin(n\alpha) \\ 5^n \cos(n\alpha) \end{bmatrix} .$$

2. Solution. $B = S^{-1}AS = \begin{bmatrix} 4-3i & 0 \\ 0 & 4+3i \end{bmatrix}$ is diagonal, where $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ contains eigenvectors as columns. Since $B^n = S^{-1}A^nS = \begin{bmatrix} (4-3i)^n & 0 \\ 0 & (4+3i)^n \end{bmatrix}$. We get $A^n\vec{v} = S \begin{bmatrix} (4-3i)^n & 0 \\ 0 & (4+3i)^n \end{bmatrix} S^{-1}\vec{v} = \begin{bmatrix} \frac{1}{2i}[(4+3i)^n - (4-3i)^n)] \\ \frac{1}{2}[(4+3i)^n - (4-3i)^n)] \end{bmatrix}$.

42) a) We can write the system as $T(x,y) = (x - ky, y + k(x - ky)) = (x - ky, kx + (1-k)y)$. The corresponding matrix is $A = \begin{bmatrix} 1 & -k \\ k & (1-k^2) \end{bmatrix}$.
b) All eigenvalues of the matrix A satisfy $|\lambda| = 1$ because $\det(A) = 1$, $\text{tr}(A) = 2 - k^2 < 2$.

28) If $x(t+1) = Ax$ has a stable origin, then the eigenvalues of A satisfy $|\lambda| < 1$. The matrix A has the eigenvalues $\lambda + 2$. They can no more satisfy $|\lambda| < 1$ and the origin is unstable.

38) a) $Aw + b = w$ means or $(A-1)w = -b$ or $w = (A-1)^{-1}b$.
The affine transformation can be modeled by a linear transformation $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. The eigenvalues of B are the eigenvalues of A and 1. If all eigenvalues of A are smaller then 1, then $T^n\vec{v}$ converges to $\vec{w}$.

Section 8.1: Symmetric matrices

10) The eigenvalues are 0 and 9. An eigenvector to 9 is $[1,-2,2]$, eigenvectors to 0 are $[1,-2,2]$ and $[-2,0,1]$. Normalize them to get an orthonormal eigenbasis.

24) The characteristic polynomial is $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2$. We can find eigenvectors $[1,0,0,1], [0,1,1,0]$ to the eigenvalues 1 and $[-1,0,0,1], [0,-1,1,0]$ to the eigenvalues -1.