**Section 7.3 : Finding the eigenvalues of a matrix**

8) The matrix is upper triangular and has therefore as eigenvalues the diagonal elements of $A$ which are $1, 2, 3$. Because the eigenvalues are different, the geometric multiplicity of each eigenvalue is 1. The eigenvalue 1 has the eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. To get the eigenvector to the eigenvalue 2, compute the kernel of $A - 2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. To get the eigenvector to the eigenvalue 3, compute the kernel of $A - 3 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ which is row reduced the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ and has the kernel $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

16) The characteristic polynomial is $\lambda(\lambda^2 + 1)$. The matrix has the eigenvalues $0, i, -i$. The eigenvector to the eigenvalue 0 is the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. It is now enough to compute the eigenvector $\vec{v}$ to the eigenvalue $i$ because the conjugate vector $\bar{\vec{v}}$ is the eigenvector to $-i$. The eigenvector to $i$ is $\begin{bmatrix} 1 \\ i - 1 \\ 2 \end{bmatrix}$, the eigenvector to $-i$ is $\begin{bmatrix} 1 \\ -i - 1 \\ 2 \end{bmatrix}$.

20) The eigenvalues 1 has algebraic multiplicity 2. The eigenvalue 2 has algebraic multiplicity 1. The geometric multiplicity of the eigenvalue 2 does not depend on $a, b, c$. Look at $\ker(A - 1) = \ker(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 2 \end{bmatrix}$. You see that this is two dimensional if $a = 0$ and one dimensional if $a \neq 0$. Therefore, for $a = 0$, the geometric multiplicity of 1 is 2 and we have an eigenbasis. For $a \neq 0$, the geometric multiplicity of 1 is 1 and we have no eigenbasis.

28) This matrix is a Jordan normal block. Since $J - k$ has rank $n - 1$, the kernel is one dimensional. The only eigenvalues of $J$ is $k$. It has algebraic multiplicity $n$ and geometric multiplicity 1.

42) From $C(t + 1) = 0.8C(t) + 10$, we get

$$A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix}.$$

This matrix has an eigenvector $\vec{v}_1 = [50, 1]$ to the eigenvalue 1 and an eigenvector $\vec{v}_2 = [1, 0]$ to the eigenvalue 0.8. The vector $\vec{v} = [0, 1]$ can be written as $\vec{v} = \vec{v}_1 - 50\vec{v}_2$ and so $A^n\vec{v} = A^n\vec{v}_1 - 50A^n\vec{v}_2 = \vec{v}_1 - 50(0.8)^n = [50(1 - 0.8^n), 1]$. Therefore $C(t) = 50(1 - 4^t/5^t)$. The function is 0 at $t = 0$, increases monotonically for increasing $t$ and reaches asymptotically, the value 50.

48) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda_1 = (1 + \sqrt{5})/2$ with eigenvector $\vec{v}_1 = [\lambda_1, 1]$ and the eigenvalue $\lambda_2 = (1 - \sqrt{5}/2$ with eigenvector $\vec{v}_2 = \lambda_2, 1]$. Because $[1, 0] = (\vec{v}_1 - \vec{v}_2)/\sqrt{5}$, we know

$A^n\vec{v} = (A^n\vec{v}_1 - A^n\vec{v}_2)/\sqrt{5} = (\lambda_1^n\vec{v}_1 - \lambda_2^n\vec{v}_2)/\sqrt{5}$. So $j(n) = (\lambda_1^n - \lambda_2^n)/\sqrt{5}, a(n) = j(n+1)$. In the limit $n \to \infty$, the term $\lambda_2^n$ vanishes and we have $j(n+1)/j(n) \to \lambda_1$, the golden ratio.

## Section 7.4 : Diagonalization

14) The matrix has three different eigenvalues and is therefore diagonalizable. The eigenvectors are $[1,0,0], [-1,1,0], [0,-1,1]$. The matrix $S$, diagonalizing $A$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

18) The matrix has the eigenvalues $0, 1, 2$, which are different, so $A$ is diagonalizable. The eigenvectors are $[1,0,1], [0,1,0], [-1,0,1]$. The matrix $S$ diagonalizing $A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

20) To investigate the geometric multiplicities of the eigenvalue 1, look at the kernel of $A - 1 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$. If $a$ is zero, the geometric multiplicity is 2 otherwise, the geometric multiplicity is 1. To investigate the geometric multiplicities of the eigenvalue 2, look at the kernel of $A - 2 = \begin{bmatrix} -1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$. The geometric multiplicity is always 1. There is an eigebasis for $a = 0$.

26) The determinant is the product of the eigenvalues. If the determinant is negative, there must be an odd number of negative eigenvalues and an odd number of positive eigenvalues. Since 0 is odd, there is at least one positive eigenvalue.

36) The first matrix has the eigenvalues $2, 3$ and both have the same trace and determinant so that both matrices have the same eigenvalues. Because both matrices are diagonalizable, they are similar.

54) If they were, also $A^2$ and $B^2$ were similar but $A^2 = 0$ and $B^2 \neq 0$. The matrices are not similar.

58) This is the Cayley-Hamilton theorem. Since it is true for the diagonalized matrix $B$, we apply $S$ to the left and $S^{-1}$ to the right of the identity $f_A(B) = 0$ to get also $f_A(A) = 0$.

56) The characteristic polynomials of both sides in the hint are the same. Because of the product property of the characteristic polynomial, the characteristic polynomial of the left hand side is the one of $AB$, while the characteristic polynomial of the right hand side is the one of $BA$.

## Section 7.5 : Complex Eigenvalues

8) $(\cos(3\theta) + i\sin(3\theta) = e^{3i\theta} = (e^{i\theta})^3 = (\cos(\theta) + i\sin(\theta))^3$. So: $\boxed{\cos(3\theta)}$ + i $\boxed{\sin(3\theta)}$ $= (\cos(\theta) + i\sin(\theta))^3 = \boxed{\cos^3(\theta) - \cos(\theta)\sin^2(\theta)}$ + i $\boxed{(\cos^2(\theta)\sin(\theta) - \sin^3(\theta))}$.

12) The characteristic polynomial of the complex conjugate $\overline{A}$ is $\overline{f}_A(\lambda)$, the polynomial, where all coefficients are conjugated. But this is the same as the characteristic polynomial of $A$ because $A$ is real. Since $f_A(\lambda) = \prod_i(\lambda - \lambda_i) = \overline{f}_A(\lambda) = \prod_i(\lambda - \overline{\lambda_i})$, the fundamental theorem of algebra assures that the set $\{\overline{\lambda_i}\}$ is the same as the set $\{\lambda_i\}$.

24) The characteristic polynomial is $\lambda^3 - 3\lambda^2 + 7\lambda - 5$ which has three roots $1, 1 + 2i, 1 - 2i$.

There is an eigenvector $[1, 1, 1]$ to the eigenvalue 1. The eigenvector to $1+2i$ is $[-3+4i, 5+10i]$, the eigenvector to $1 - 2i$ is $[-3 - 4i, 5 - 10i]$.

32) The transition matrix is

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

If you use the power method, you will see that $(0.4, 0.35, 0.25)$ is the stable equilibrium. Indeed $(0.4, 0.35, 0.25)$ is an eigenvector of A. Instead of using the power method, you could have computed the eigenvector to the eigenvalue 1. Note that these matrices always have an an eigenvalue 1, because $A^T$ has the property that the sum of the columns is 1 and $A^T$ therefore has the eigenvector $(1, 1, 1)$. Because $A^T$ and $A$ have the same eigenvalues, also A has the eigenvalue 1. The corresponding eigenvector is the stable equilibrium.

38) a) $C, C^2, C^3, A^4$ are again permutation matrices.

b) The matrix has the characteristic polynomial $\lambda^4 - 1$ and the eigenvalues $1, i, -1, -i$.

c) The circulant matrix $M$ is the sum $aC^0 + bC^1 + cC^2 + dC^3$ and has the same eigenvectors than $C$. The eigenvalues of $M$ are $a + b\lambda + c\lambda^2 + d\lambda^3$.