

Section 5.2 Coordinates

2) Normalize $\vec{w}_1 = \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}$, then form $\vec{u}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \vec{v}_2$ and normalize $\vec{w}_2 = \vec{v}_2/7$.

$$14) \vec{w}_1 = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}.$$

$$\vec{w}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}.$$

$$16) Q = \begin{bmatrix} 6/7 & 2/7 \\ 3/7 & -6/7 \\ 2/7 & 3/7 \end{bmatrix}. R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

34) To compute the kernel, we bring the matrix A into row reduced echelon form: $\text{rref}(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, find a basis for the kernel and then make it orthogonal using Gram-Schmidt.

We end up with $\vec{w}_1 = [1, -2, 1, 0]^T/\sqrt{6}$ and $\vec{w}_2 = [2, -1, -4, 3]^T/\sqrt{30}$.

40) The matrix R will be diagonal.

42) The product is the area of the parallelepiped spanned by the two vectors.

Section 5.3 Projection

6) Yes, we have to show that if $A^T A = I_n$, then also $AA^T = B^T B = I_n$, where $B = A^T$.

From $A^T A = I_n$, we get $A^T = A^{-1}$ which can be read as $B = (B^T)^{-1}$. Taking inverses shows $B^{-1} = B^T$. By multiplying both sides from the right with B , we end up with $B^T B = I_n$.

(It is useful to remember from this that also the rows of an orthogonal matrix form an orthonormal basis.)

8) a) No, take just the example on the same page, where $A^T A$ is I_2 while AA^T is a projection matrix.

b) Yes, it is the case because we have seen in problem 6) that $B = A^T$ is also $AA^T = I_n$.

16) Yes: $A = A^T$ implies $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$.

Note that if A, B are different symmetric matrices, then AB is not necessary symmetric any more.

20) The two vectors are not yet orthonormal, but a Gram-Schmidt orthonormalisation gives

to normal vectors which can be used to define $A = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}$ and get the projection

$$P = AA^T = \begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix} / 50.$$

40) The two vectors are not yet orthonormal. We do Gram-Schmidt orthogonalization:

$$\vec{w}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix} / 10$$

The matrix of the orthogonal projection is QQ^T , where

$$Q = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}.$$

which is

$$\begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix} / 50.$$

18) a) $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

b) Actually, A^2 is symmetric, because $(A^2)^T = (AA)^T = (-A^T)(-A^T) = (A^T)^2$.

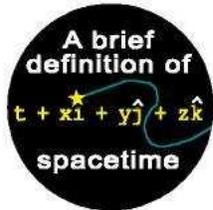
44) We can write any quaternion matrix as $M = p1 + qi + sj + rk$, where

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

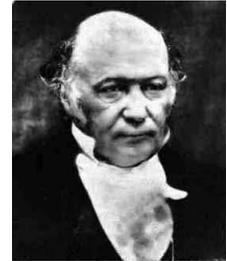
From this is obvious that M is a linear space of dimension 4. Because $i \cdot j = -i \cdot j = k$, $j \cdot k = -k \cdot j = i$, $k \cdot i = -i \cdot k = j$, $i \cdot i = j \cdot j = k \cdot k = -1$, the product of quaternion matrices is a quaternion matrix. We compute $M^T M = (p^2 + q^2 + r^2 + s^2)I_4$ and see from this that M is invertible if $(p^2 + q^2 + r^2 + s^2)$ is not zero. In the other case, we have the zero matrix. The inverse is in

M too because $M^{-1} = M^T / (p^2 + q^2 + r^2 + s^2)$ and $M^T = -M + pI_4$ is a quaternion matrix too. Quaternions do not commute because $ij = -ji$.

ROTATION WITH QUATERNIONS. One can calculate with quaternions directly without the matrix representation. One often writes $q = (s, \vec{p}) = s + ip_1 + jp_2 + kp_3$. Quaternions are useful to compute rotations in space. If one wants to rotate a vector \vec{v} in space by an angle ϕ around an axis which contains the unit vector \vec{u} , one can form the quaternions $q = (\phi, \vec{u})$ and $p = (0, \vec{v})$ and form the new quaternion $p' = qpq^{-1}$. It has the form $p' = (0, \vec{v}')$, where \vec{v}' is the rotated vector. This algebraic manipulation is useful in physics or computer graphics.



DISCOVERY. Quaternions were discovered by William Rowan Hamilton while walking along the Royal Canal. He was so excited about his "invention" that he wrote the properties $i^2 = j^2 = k^2 = ijk = -1$ of the quaternions into the stone of the Brougham bridge. Quaternions are also called **hypercomplex numbers**. They are not only used in computer graphics, also physicists find them handy.



Section 5.4 Orthogonality and least squares

2) The kernel of A^T is equal to the orthogonal complement of the image of A . Since the later is a plane, the kernel of A^T is a line. To find the kernel of $B = A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, we do row reduction and get $\text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ from which we see that $[1, -2, 1]^T$ is a basis for the kernel.

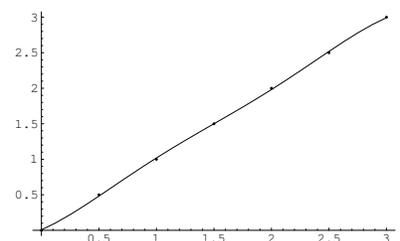
10) a) Consistent means that we have a solution $A\vec{v} = \vec{b}$. The hint leads the path: split up this solution $\vec{v} = \vec{v}_h + \vec{v}_0$, where \vec{v}_h is in the kernel of A and \vec{v}_0 in the complement. Now, $A\vec{v}_h = 0$ implies $A\vec{v}_0 = \vec{b}$ also and we found the solution.

b) If there were two solutions \vec{x}_0, \vec{y}_0 in the orthogonal complement of the kernel, then the element $\vec{v} = \vec{x}_0 - \vec{y}_0$ in the orthogonal complement of the kernel satisfies $A\vec{v} = \vec{0}$. But because \vec{v} is both in the kernel and the orthogonal complement, it is $\vec{0}$ and $\vec{x}_0 = \vec{y}_0$.

c) This follows from Pythagoras and the relation $\vec{x}_1 = \vec{x}_0 + \vec{x}_h$.

22) Use the routine formula $(A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. The error is 0, the system is actually consistent.

34) $f(t) = 1.5 - 1.53669 \cos(t) + 0.0431477 \cos(2t) + 0.108974 \sin(t) + 0.302692 \sin(2t)$. It was of course ok to use technology here.



40) From $D = ka^n$ we get $\log(D) = \log(k) + n \log(a)$. We fit the data points $(\log(a_i), \log(D_i))$ with respect to the functions $1, t$ and get the best linear fit $f(s) = 0.00048689 + 1.49982s \sim 3s/2$. This means that $g(t) = e^{3t/2}$ is the best fit.

$D = a^{3/2}$ means $D^2 = a^3$ which is Keplers third law.

The constant K is one because units have been chosen so that for the earth the constant is $K = 1$.

16) Assume A is a $m \times n$ matrix so that A^T is a $n \times m$ matrix Apply dimensions to $\text{im}(A)^\perp = \ker(A^T)$ gives $m - \text{rank}(A) = m - \text{rank}(A^T)$, where we have used that $\dim \ker(A^T) + \dim \text{ran}(A^T) = m$.

18) The rank of $A^T A$ is smaller or equal then the rank of A . Because the kernel of A^T is perpendicular to the image of A , the rank of $A^T A$ is indeed equal to the rank of A . and because the ranks of A and A^T agree also the rank of $A^T A$ is equal to the rank of AA^T .