Section 3.3 Dimension.

22) $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The first and third columns are Pivot columns, so that $1, 3$ of the original are a basis of the image of A. The kernel is 3-dimensional. We can introduce free variables s, t, u for columns $2, 4, 5$. If $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ v \\ w \end{bmatrix}$, then $w = u, v = t, z = t - u, y = s, x = -2s - 3t$, so that $s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a general element in the kernel.

24) Form

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where A contains the given vectors as columns. The first and third columns are pivot columns. Therefore, the first and third vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ 6 \\ 3 \end{bmatrix}$ span the subspace.

32) We look for the kernel of the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ which is already in row reduced form. The first two columns are pivot columns. Attach free variables to the last two columns so that $w = t, z = s, x = s - t, y = -2s - 3t$, so that a general element in the kernel is $\vec{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

38) a) Because the image has dimensions $0, 1, 2$ or 3, the kernel has by the dimension formula dimensions $5, 4, 3$ or 2.

b) The image of T has maximal 4 dimensions (look at the matrix in reduced row echelon form, there can be maximal 4 leading 1 and therefore maximally 4 pivot columns). The possible values of the rank of T are $0, 1, 2, 3, 4$.

52) Write down a new matrix, which continas the rows of A as the columns. This is called the **transpose** of A. The first and third column of the transposed matrix A^T are Pivot columns so that the first and third row of A form a basis of the row space.

36*) No, this is not possible by the dimension formula. The dimensions of the image and kernel have to add up to 3.

56*) The hint gives the solution away already. Assume these vectors are linearly dependent, then one could have

$$c_0\vec{v} + c_1 A\vec{v} + ... + c_{m-1}A^{m-1}\vec{v} = \vec{0} \, .$$

Multiplying both sides with A^{m-1} using $A^m = 0$ shows that $c_0 = 0$. We are left with

$$c_1 A\vec{v} + ... + c_{m-1}A^{m-1}\vec{v} = \vec{0} \, .$$

Multiply both sides with A^{m-2} to see that $c_1 = 0$. etc.

Section 3.4 Coordinates

2) The vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is perpendicular to the plane so that we can take

$$S = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

as the coordinate transformation. The inverse is

$$S^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ -2 & 2 & -2 \\ 1 & 2 & 1 \end{bmatrix} /6$$

and $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.

14) $A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}$. $S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

The matrix in the coordinates of the new basis is $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

16) a) Note that the two first basis vectors are in the plane while the third is perpendicular to the plane. Therefore, in that basis, the transformation is given by the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The matrix in the standard basis is $A = SBS^{-1}$, where $S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$. We get

$$A = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix} /7 \, .$$

22) The vector $\vec{x}$ is $-\vec{v} + 2\vec{w}$. Flip the vector $\vec{v}$ and add two times the vector $\vec{w}$ in the picture.

26) Yes, the transformation is linear because it is given by a matrix S^{-1}.

32)* a) The hint gives the solution away see 56) above.
b) $\vec{v}_1 = A^2\vec{v}$ is mapped to $A^3\vec{v} = 0$, $\vec{v}_2 = A\vec{v}$ is mapped to $A^2\vec{v} = \vec{v}_1$ and $\vec{v}_3 = \vec{v}$ is mapped to

$A\vec{v} = \vec{v}_2$. Therefore, the matrix in that basis is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

38)* Yes, they are similar. We will learn later a general method to check such things. At this stage of the course, we have maybe to experiment a bit. For example:

1) $S_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ flips both the diagonal elements as well as the side diagonal elements.

2) $S_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ flips the side diagonals as well as the signs of the side diagonals.

3) $S_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ changes the signs of the side diagonals.

Bingo! Combining 2) and 3) achieves the goal $S = S_1 S_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ just flips the side diagonals and so

$$\begin{bmatrix} a & d \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Note that $S = S^{-1}$ in this case.

Section 4.1 Linear spaces

In problems 6-11, we have always to check three properties: 1) we can add in the set and stay in the set, we can scale an element and still are in the set and 0 is in the set.

6) Invertible matrices form not a linear space because the zero matrix is not invertible.

7) The diagonal matrices form a linear space. All three properties are easily checked to be true.

8) The upper triangular 3×3 matrices form a linear space.

9) The 3×3 matrices whos entries are ≥ 0 form no linear space. If A is such a matrix, then $-A$ is not in the space.
10) All the matrices which have a given vector v in the kernel form a linear space. If we add such matrices then $(A + B)v = Av + Bv = 0$ etc.

11) The 3×3 matrices in row reduced echelon form form no linear space. If you add two such matrices, their sum is no more in row reduced echelon form in general. For example, adding two identity matrices is no more a matrix in rref.

36) Write $A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & k \end{bmatrix}$. The matrix $AB - BA$ is $\begin{bmatrix} 0 & b & c \\ -e & 0 & 0 \\ -h & 0 & 0 \end{bmatrix}$. In order that is zero, we must have $b = c = e = h = 0$.

48) a) A general polynomial in P_4 which is even satisfies $ax^4 + bx^2 + c$. The dimension is 3.

b) A general polynomial in P_4 which is odd satisfies $ax^3 + bx$. The dimension is 2.
58) a) follows from $\cos^2(x) + \sin^2(x) = 1$.
b) we know $f'' = 0, f'(0) = 0, f(0) = 0$ which implies $f == 0$.

c) $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$ is a sum of elements in V and therefore in V.

44) Write $S = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Now $\begin{bmatrix} \vec{u} & \vec{v} & 0 \end{bmatrix}$. Since this is also $AS = \begin{bmatrix} Au & Av & Aw \end{bmatrix}$, we must have $Aw = 0, Av = v, Au = u$. This means that u, v must be in the plane and w must be perpendicular to that plane. We have 2 parameters to chose a vector in the plane and 4 parameters to chose two vectors in the plane. There is additional parameter to chose a vector perpendicular to the plane. The answer is 5.
12) Is no linear subspace.

13) Is also no linear subspace.

14) This is a linear subspace.

15) Als this is a linear subspace.