

Section 2.4 Matrix Multiplication.

$$4) \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$$

$$14) BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}, BD = [6], CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, EB =$$

$$[5, 10, 15]. \text{ If you want to pair matrices with itself, we can also form } A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} C^2 =$$

$$\begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}, E^2 = [25].$$

$$28) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is a possibility.}$$

Remark for later: Actually, $A^2 = 0$ implies that A is similar to that transformation (see later). The image of A is contained in the kernel of A and must be one dimensional by the dimension formula (seen later). In a basis, where one of the basis vectors is in the kernel, the transformation looks like that.

$$40) AB = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}. A = (AB)B^{-1} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}.$$

$$76) \text{ a) } A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

$$\text{b) } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\text{c) } AB = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}.$$

$$\text{d) } M = A^{-1}BA = \begin{bmatrix} 2/3 & 1/3 & -2/9 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}.$$

$$48) T : P_2 \rightarrow P_2, P_1 \rightarrow P_3 \rightarrow P_0 \rightarrow P_1.$$

$$L : P_0 \rightarrow P_0, P_3 \rightarrow P_3, P_1 \rightarrow P_2 \rightarrow P_1.$$

a) Inverse rotation T^{-1} . b) Reflection

c) Since $T^3 = I$, we know that T^2 is the inverse rotation T^{-1} .

d) $T \circ L : P_0 \rightarrow P_1, P_1 \rightarrow P_2, P_2 \rightarrow P_3, P_3 \rightarrow P_0$.

$L \circ T : P_0 \rightarrow P_2, P_1 \rightarrow P_3, P_2 \rightarrow P_1, P_3 \rightarrow P_0$.

e) $P_0 \rightarrow P_2 \rightarrow P_3 \rightarrow P_0, P_1 \rightarrow P_1$ is a rotation.

$$66) A \text{ is invertible if and only if } A_{11} \text{ and } A_{22} \text{ are both invertible. The inverse is } \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}.$$

One can find the inverse by writing out $BA = 1$ as four equations for the four unknown entries of B .

Section 3.1 Image and Kernel

10) We bring the matrix in row reduced echelon form. Starting to clean out the second column $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and end up with $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. If we write down the

system $B \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we get $w = 0, z = s, y + 2s = 0, x - s = 0$, where s is a free variable introduced in the third column, the only one without leading one. The kernel is the line

spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

22) $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The image is spanned by $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$,

the two first columns of A . The kernel is spanned by $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

34) Take the projection on the plane $-x + y + 2z = 0$ for example. There are other possibilities.

44) Under a row reduction, the image changes but not the kernel.

For a) we can take as a counter example the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The kernel of A is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the image of A is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. the image of $\text{rref}(A)$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b) The kernel does not change under row reduction. Check that for switching rows, scaling rows or adding rows to an other row.

54) If there was no error, then $Hw = 0$ because $H(Mx) = 0$ if there was no error. The columns of H encode, which of the bits are switched. We can look match $H(M(x + e))$ with the columns and get the offending bit.

38) a) $Ax = 0$ implies $A^2x = 0$ so that $\ker(A^2)$ contains $\ker(A)$ and more generally, $\ker(A^n)$ contains $\ker(A^{n-1})$. The inclusion can be strict, like with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

b) $\text{im}(A^2)$ is a subset of $\text{im}(A)$ and more generally $\text{im}(A^n)$ is a subset of $\text{im}(A^{n-1})$. The inclusion can be strict, like with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ where the image of A^2 is contained in the one dimensional $\text{im}(A)$.

48)

a) $w = Av$ and $Aw = A^2v = Av = w$ so that $Aw = w$.

b) If $\text{rank}(A) = 2$ then $A^2 = A$ implies $A = 1$.

If $\text{rank}(A) = 0$, then $A = 0$.

c) If $\text{rank}(A) = 1$, then Ax is a multiple of a vector w . Assume $Au = 0$. Since by a), $Aw = w$,

and $Au = 0$, the two vectors w and u span the plane. A general vector is of the form $su + tw$. We have $Ax = tw$ so that A is a projection along u onto the line spanned by w .

Section 3.2 Basis

6) The intersection is a linear subspace, the union not.

18) Linear dependent.

24) Finding the basis of the orthogonal complement is identical to find a basis for the kernel of

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

We can take $\begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ for example.

28) The columns are linearly independent and form a basis.

48) It is the kernel of the matrix $A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. It is the image of a matrix $B = \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$.

36) The image vectors are linearly dependent too.

38)

a) See fact 3.2.6.

b) If they would not span V , we could add an other vector and still have a linear independent set.

c) Just take the vectors v_j as the columns of the matrix.