Section 2.1 Linear transformations. (6,14,28,42,44,34*,(24-30)*)

6) Row picture: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are the rows of the matrix, we are looking for. With

$A == \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ one has $A\vec{x} = T(\vec{x})$. Indeed, the transformation is linear.

14) a) In order to see whether a matrix is invertible, we row reduce.

$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ $\begin{bmatrix} 2 & 3 \\ 0 & 15 - 2k \end{bmatrix}$. The matrix is not invertible, if and only if $2k = 15$.

b) All entries are integers if the solution $\vec{x} = (x, y)$ to the equation $A\vec{x} = \vec{b}$ with integer vector $\vec{b} = (a, b)$ has integers. If we row reduce the augmented matrix $B = \begin{bmatrix} 2 & 3 & a \\ 5 & k & b \end{bmatrix}$ we end up with

$\text{rref}(B) = \begin{bmatrix} 1 & 0 & (3b - ak)/(2k - 15) \\ 0 & 1 & (5a - 2b)/(2k - 15) \end{bmatrix}$ which shows that $2k - 15$ must be either a fraction $1/n$. That means $k = (15 + 1/n)/2$, where n is a nonzero integer.

28) The face gets streched in the y coordinates by a factor 2.

42) a) The image of $\vec{0}$ is $\vec{0}$. The image of the 3 basis vectors e_i are the columns of A. The image of the other 4 vectors can be obtained by linearity.

b) To find the points which are maped to zero, we must have $-x/2 + y = 0$ and $-x/2 + z = 0$. Choosing y freely, say s, we have $x = 2s, y = z = s$. So, all the points on the line $\begin{bmatrix} 2s \\ s \\ s \end{bmatrix}$ are maped to zero.

P.S. We will later call the set of all this vectors the **kernel** of A.

44) Yes, the transformation is linear. The matrix is obtained by applying the map to the fundamental vectors e_i:
$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

34*) The trick is to draw the images of the basis vectors and put that as the columns of the matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

24) is a rotation by 90 degrees in the counter clockwise direction,

25) is a scaling (dilation) by a factor 2,

26) is a reflection at the line $x = y$,

27) is a reflection at the y axes,

28) streches by a factor 2 in the y direction.

29) is a reflection at the origin,

30) is a projection onto the y-axes.

Section 2.2 Linear transformations. (2.2: 4,8,10,30,34,47*,50*)

4) It is a rotation dilation. A dilation by a factor $\sqrt{2}$ followed by a rotation by $\pi/2$.

8) A shear parallel to the x axes.

10) Just take the basis vectors and map them under the map $T(\vec{x}) = (\vec{u} \cdot \vec{x})\vec{u}/(25)$. The matrix is $\begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$.

30) It is a projection on the line containing the vector $(1, 2)$.

34) For the transformation which is a projection onto a line, all three basis vectors e_i should be maped into the same vector. That is the case for B only. For a reflection in a line, the map should have the property that $T(T(x)) = x$. This is only the case for transformation E.

47) Define $e_1(t) = R(t)e_1, e_2(t) = R(t)e_2$, where $R(t)$ is the rotation by an angle t. We have $f(t) = T(e_1(t)) \cdot T(e_2(t))$.

a) If $T(x, y) = (ax + by, cx + dy)$, then $f(t) = (a\cos(t) + b\sin(t), c\cos(t) + d\sin(t)) \cdot (-a\sin(t) + b\cos(t), -c\sin(t) + d\cos(t)) = (a^2 + b^2 - c^2 + d^2/2)\sin(2t) + (ab + cd)\cos(2t)$ which is continuous.
b) $f(\pi/2) = -f(0)$ follows from a).
c) This is the intermediate value theorem.
d) If $f(t) = 0$, then the two vectors $T(e_1(t)), T(e_2(t))$ are perpendicular.

This exercice shows that we can turn the coordinate system in such a way that the images of the vectors are perpendicular to each other. If T is noninvertible, then we can turn the coordinate sysetm such that one vector is maped to the zero vector.

50) Assume, we have two perpendicular vectors $\vec{v}_1$, $\vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are orthogonal. The unit circle can be written as $cos(t)v_1 + \sin(t)v_2$. The image is $\cos(t)w_1 + \cos(t)w_2$.

An other approach would be to verify that transformations like rotations, dilations, diagonal transformations and reflections map ellipses into ellipses and that one can write any transofor-mation as a composition of such transformations.

Section 2.3 Linear transformations. (2.3: 10,20,30,40*,42*)

10)

Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 3 & | & 0 & 1 & 0 \\ 1 & 3 & 6 & | & 0 & 0 & 1 \end{bmatrix}$. gives $\begin{bmatrix} 1 & 0 & 0 & | & 3 & -3 & 1 \\ 0 & 1 & 0 & | & -3 & 5 & 2 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{bmatrix}$.

20) Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 3 & 3 & | & 1 & 0 & 0 \\ 1 & 4 & 8 & | & 0 & 1 & 0 \\ 2 & 7 & 12 & | & 0 & 0 & 1 \end{bmatrix}$, we end up with

$\begin{bmatrix} 1 & 0 & 0 & | & -8 & -15 & 12 \\ 0 & 1 & 0 & | & 4 & 6 & -5 \\ 0 & 0 & 1 & | & -1 & -1 & -1 \end{bmatrix}$.

30) The map is never invertible.

40) If the i'th and j'th row are the same, then the vector $\vec{u} = e_i - e_j$ is maped to the zero vector. This means that the vectors $\vec{v}$ and $\vec{v} + \vec{u}$ are maped to the same point $\vec{z}$. The inverse of that vector $\vec{z}$ is not defined since it both would have to be $\vec{v}$ or $\vec{u}$. In our case, both the vectors $\vec{v}$ and $\vec{u} = \vec{0}$ are maped to $\vec{z}$.

42) Yes, permutation matrices are invertible, because we can invert the permutation. For example, the inverse of the permutation matrix which belongs to the permutation $(1, 2, 3) \to (2, 3, 1)$ is the permutation matrix which belongs to the permutation $(1, 2, 3) \to (3, 1, 2)$.

An other way to see that a permutation matrix is invertible is to row reduce it. It always can be row reduced to the identity matrix by switching rows alone.

Section 2.4 Matrix Multiplication. 2.4: 14,28,48*,72*

14) $BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$, $BD = [6]$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, $DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB =$

$[5, 10, 15]$. If you want to pair matrices with itself, we can also form $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $C^2 =$

$\begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $E^2 = [25]$.

28) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a possibility.

Remark for later: Actually, $A^2 = 0$ implies that A is similar to that transformation (see later). The image of A is contained in the kernel of A and must be one dimensional by the dimension formula (seen later). In a basis, where one of the basis vectors is in the kernel, the transformation looks like that.

40) $AB = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. $A = (AB)B^{-1} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$.

72) Trying $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ gives $A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$. In order that $a = 2a^2$, we must have

$a = 1\sqrt{2}.$