Section 10.4 Solutions

1. Notice that $T(x,y) = xy$ satisfies Laplace's equation, since $\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial t^2} = 0$. So if the center of the disk is at (x_0, y_0) then the temperature there could be $x_0 y_0$.

But can we come up with another solution? Let's assume that the disk is located in the square $[0, 1] \times [0, \pi]$, and let the temperature function on the disk be $s(x,y)$ subject to the constraint that $s(x,y) = xy$ on the boundary and $\frac{\partial^2 s}{\partial t^2} + \frac{\partial^2 s}{\partial y^2} = 0$. Then $T(x,y) = s(x,y) - xy$ gives the temperature of a disk such that the temperature is 0 on the boundary. Now extend $T(x,y)$ to be zero on the rest of the square; then this extended function satisfies Laplace's equation as well as the boundary conditions $T(x,0) = T(x,\pi) = T(0,y) = T(\pi,y) = 0$. Note that we can do this because T is continuous, as it is zero on the boundary of the disk. Given these boundary conditions, all methods only come up with one solutions for $T(x,y)$; namely $T(x,y) = 0$. Since $T(x,y) = s(x,y) - xy$, we weren't able to come up with any temperature function $s(x,y) \neq xy$. This doesn't mean that no such $s(x,y)$ exists, but it at least makes $x_0 y_0$ a reasonable answer for the temperature of the center of the disk.

2. If $s(x,y)$ satisfies Laplace's equation with the boundary conditions

$$s(x,0) = 0 \qquad s(0,y) = 0$$
$$s(x,\pi) = 0 \qquad s(\pi,y) = \begin{cases} y & y \le \frac{\pi}{2}, \\ \pi - y & y \ge \frac{\pi}{2} \end{cases}$$

and $h(x,y)$ satisfies Laplace's equation with boundary conditions

$$h(0.y) = 0 \qquad s(x,0) = 0$$
$$h(\pi,y) = 0 \qquad s(x,\pi) = \begin{cases} x & x \le \frac{\pi}{2}, \\ \pi - x & x \ge \frac{\pi}{2} \end{cases}$$

then $T(x,y) = s(x,y) + h(x,y)$ satisfies Laplace's equation with

$$T(x,0) = 0 \qquad s(0,y) = 0$$
$$T(x,\pi) = \begin{cases} x & x \le \frac{\pi}{2}, \\ \pi - x & x \ge \frac{\pi}{2} \end{cases} \qquad s(\pi,y) = \begin{cases} y & y \le \frac{\pi}{2}, \\ \pi - y & y \ge \frac{\pi}{2} \end{cases}$$

We would now like to find s and h. From page 20, we know that

$$s(x,y) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \frac{\sinh(2m+1)x}{\sinh(2m+1)\pi} \sin(2m+1)y$$

switching x and y, we see that

$$h(x,y) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \frac{\sinh(2m+1)y}{\sinh(2m+1)\pi} \sin(2m+1)x$$

So that

$$\begin{aligned}T(x,y) &= s(x,y)+h(x,y)\\ &= \sum_{m=0}^{\infty}\frac{4(-1)^m}{\pi(2m+1)^2\sinh(2m+1)\pi}\left(\sinh(2m+1)x\sin(2m+1)y+\sinh(2m+1)y\sin(2m+1)x\right)\end{aligned}$$

3. We know that

$$u(x,t)=\sum_{n=0}^{\infty}(a_n\sin nt+b_n\cos nt)\sin nx$$

is a general solution to the differential equation $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial t^2}$ subject to the constraints $u(0,t)=u(\pi,t)=0$. If the string is initially undisturbed, we have that $u(x,0)=0\Rightarrow b_n=0$ for all n. Furthermore, if

$$\frac{\partial u}{\partial t}(x,0)=\begin{cases}x, & x\le\frac{\pi}{2}\\ \pi-x, & x\ge\frac{\pi}{2}\end{cases}=\Theta(x)$$

then $\frac{\partial u}{\partial t}(x,0)=\sum_{n=0}^{\infty}na_n\cos nt\sin nx\big|_{t=0}=\sum_{n=0}^{\infty}na_n\sin nx=\Theta(x)$. We now need to find a Fourier sine series for $\Theta(x)$. Extending to the interval $[-\pi,\pi]$, we get

$$\Theta(x)=\begin{cases}x, & -\frac{\pi}{2}\le x\le 0\\ -\pi-x, & -\pi\le x\ \le -\frac{\pi}{2}\end{cases}.$$

Since $\Theta(x)=\sum_{m=0}^{\infty}\frac{4(-1)^m}{\pi(2m+1)^2}\sin(2m+1)x$, we obtain $(2m+1)a_{2m+1}=\frac{4(-1)^m}{\pi(2m+1)^2}\Rightarrow a_{2m+1}=\frac{4(-1)^m}{\pi(2m+1)^3}$, $a_{2m}=0$, and

$$u(x,t)=\sum_{m=0}^{\infty}\frac{4(-1)^m}{\pi(2m+1)^3}\sin(2m+1)t\sin(2m+1)x.$$

4. Suppose that $f(y)$ and $g(y)$ are twice-differentiable functions and that $u(x,t)=f(x+t)+g(x-t)$ satisfies the equation $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$. We see that this is always the case, as

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 f(x+t)}{\partial x^2}+\frac{\partial^2 g(x-t)}{\partial x^2}=f''(x+t)+g''(x-t)\\ \frac{\partial^2 u}{\partial t^2}&=\frac{\partial^2 f(x+t)}{\partial t^2}+\frac{\partial^2 g(x-t)}{\partial t^2}=f''(x+t)+9-1)^2g''(x-t)=\frac{\partial^2 u}{\partial x^2}\end{aligned}$$

The condition $u(0,t)=u(\pi,t)=0$ then requires

$$\begin{aligned}u(o,t)&=f(t)+g(-t)=0\Rightarrow f(y)=-g(-y)\\ u(\pi,t)&=f(\pi+t)+g(\pi-t)=0\Rightarrow f(\pi+t)-f(t-\pi)=0\\ &\Rightarrow f(2\pi+y)=f(y)\end{aligned}$$

From the initial condition $\frac{\partial u}{\partial t}(x,0)=0$, we get

$$\frac{\partial u}{\partial t}=\frac{\partial f(x+t)}{\partial t}-\frac{\partial f(t-x)}{\partial t}=f'(x+t)-f'(t-x)$$
$$\Rightarrow f'(x)-f'(-x)=0\Rightarrow f(x)+f(-x)=k$$

Since $f(0O=0$, $k=0$, and we have $f(x)=-f(-x)$. Now,

$$u(x,0)=f(x+0)-f(0-x)=f(x)-f(-x)=2f(x)=\begin{cases}x, & x\le\frac{\pi}{2}\\ \pi-x & x\ge\frac{\pi}{2}\end{cases}$$

so as found on page 21 of the supplement,

$$f(x)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{4(-1)^n}{\pi(2n+1)^2}\sin(2n+1)x=\sum_{n=0}^{\infty}\frac{2(-1)^n}{\pi(2n+1)^2}\sin(2n+1)x$$

$$\Rightarrow u(x,t)=f(x+t)-f(t-x)=\sum_{n=0}^{\infty}\frac{2(-1)^n}{\pi(2n+1)^2}\left[\sin((2n+1)(x+t))-\sin((2n+1)(t-x))\right]$$

5. Let $T(x,y)=u(x)v(y)$. Then $\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=u''(x)v(y)+u(x)v''(y)=0$, so $\frac{u''(x)}{u(x)}=-\frac{v''(y)}{v(y)}=\text{const}=\pm\lambda^2$. There are three possibilities:

i. $u(x)=Ae\lambda x+Be^{-\lambda x}$, $v(y)=C\cos\lambda y+D\sin\lambda y$ $(\lambda\neq 0)$
ii. $u(x)=Ax+b$, $v(y)=Cy+D$ $(\lambda=0)$
iii. $u(x)=A\cos\lambda x+B\sin\lambda x$, $v(y)=Ce\lambda y+De^{-\lambda y}$ $(\lambda\neq 0)$.

We also have the boundary conditions

$$\frac{\partial T}{\partial x}(0,y)=u'(0)v(y)=0\Rightarrow u'(0)=0$$
$$\frac{\partial T}{\partial y}(x,0)=u(x)v'(0)=0\Rightarrow v'(0)=0$$
$$\frac{\partial T}{\partial y}(x,\pi)=u(x)v'(\pi)=0\Rightarrow v'(\pi)=0$$

In case (i), we have

$$u'(0)=A\lambda e^{\lambda x}-B\lambda e^{-\lambda x}\Big|_{x=0}=A\lambda-B\lambda=0\Rightarrow A=B$$
$$v'(0)=-C\lambda\sin\lambda y+d\lambda\cos\lambda y\big|_{y=0}\Rightarrow D=0$$
$$v'(\pi)=-C\lambda\sin\lambda\pi=0\Rightarrow c=0,\text{ or }\sin\lambda\pi=0$$

If $C=0$, then $v(y)=0\Rightarrow T(x,y)=0$, wich isn't very interesting. So we will take $\sin\lambda\pi=0\Rightarrow\lambda$ is an integer. So $u(x)=A\cosh\lambda x$ and $v(y)=C\cos\lambda y\Rightarrow T(x,y)=K\cosh\lambda x\cos\lambda y$, $\lambda\in\mathbf{Z}$.

In case (ii), we have $u'(0)=A=0$, $v'(0)=C=0\Rightarrow u(x)=B$, $v(y)=D\Rightarrow T(x,y)=BD=K'$. This corresponds to $\lambda=0$ in case (i).

In case (iii), we have

$$u'(0) = -A\lambda \sin \lambda x = B\lambda \cos \lambda x\big|_{x=0} = B\lambda = 0 \Rightarrow B = 0$$
$$v'(0) = C\lambda e^{\lambda y} - D\lambda e^{-\lambda y}\Big|_{y=0} = C\lambda - D\lambda \Rightarrow C = D$$
$$v'(\pi) = C\lambda e^{\lambda y} - D\lambda e^{-\lambda y}\Big|_{y=\pi} = C\lambda\left(e^{\lambda\pi} - e^{-\lambda\pi}\right) = 0 \Rightarrow C = D = 0$$

So this is the trivial case where $T(x, y) = 0$.

We will consider a general solution to be of the form

$$\sum_{n=0}^{\infty} c_n \cosh nx \cos ny$$

Since both cosine and hyperbolic cosine are even functions, the general form of the solution is even, as we would like. We can now impose the fourth boundary condition:

$$\frac{\partial T}{\partial x}(\pi, y) = \sum_{n=0}^{\infty} nc_n \sinh n\pi \cos ny = \sin 2y$$

We would like to reexpress $\sin 2y$ as a Fourier cosine series in order to solve for the coefficients c_n. Let $\Theta(y) = \begin{cases} \sin 2y, & 0 \le y \le \pi \\ -\sin 2y, & -\pi \le y \le 0 \end{cases}$. Then $\Theta(y)$ is an even function,and so it has a Fourier cosine series:

$$b_0 = \left\langle \Theta(y), \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{1}{\pi} = \int_{-\pi}^{\pi} \Theta(y)\frac{1}{\sqrt{2}}\, dy = \frac{1}{\pi}\int_0^{\pi} \sin 2y\, dy = 0$$

For $n > 0$, we see that

$$b_n = \langle \Theta(y), \cos ny \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta(y) \cos ny\, dy = \frac{2}{\pi}\int_0^{\pi} \sin 2y\, dy = \frac{2}{\pi}\int_0^{\pi} \sin(n+2)y - \sin(n-2)y\, dy$$

When $n \ne 2$, we get

$$\int_0^{\pi} \sin(n+2)y - \sin(n-2)y\, dy = -\frac{\cos(n+2)y}{n+2} + \frac{\cos(n-2)y}{n-2}\Big|_0^{\pi} = [(-1)^n - 1]\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

So if n is even, this integral gives 0, and otherwise it evaluates to $-\frac{8}{n^2-4}$. In the case $n = 2$, we get $\int_0^{\pi} \sin(n+2)y - \sin(n-2)y\, dy = \int_0^{\pi} \sin 4y\, dy = 0$, so

$$b_n = \begin{cases} 0, & n \text{ is even} \\ -\frac{8}{\pi(n^2-4)}, & n \text{ is odd} \end{cases}$$

So

$$\frac{\partial T}{\partial x}(\pi, y) = \sum_{n=0}^{\infty} nc_n \sinh n\pi \cos ny = \sum_{n=0}^{\infty} b_n \cos ny \Rightarrow c_n = \frac{b_n}{\sinh n\pi}$$

and we get

$$T(x,y) = \sum_{k=0}^{\infty} \frac{-8}{\pi((2k+1)^2-3)\sinh(2k+1)\pi} \cosh(2k+1)x \cos(2k+1)y.$$

6. We would like to find a solution to $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ for $0 \le x \le \pi$ and $0 \le y \le \pi$ such that

$$\frac{\partial T}{\partial x}(0,y) = 0 \quad \frac{\partial T}{\partial x}(\pi,y) = \sin y \quad \frac{\partial T}{\partial y}(x,0) = 0 \quad \frac{\partial T}{\partial y}(x,\pi) = 0$$

Green's Theorem states that $\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint M\,dx + N\,dy$. Let $N = \frac{\partial T}{\partial y}$ and $M = \frac{\partial T}{\partial x}$. Then

$$\begin{aligned} \iint_S \left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial y \partial x}\right) &= 0 \\ &= \oint \left(\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy\right) \\ &= \int_)^{\pi} \sin y\,dy = 2 \end{aligned}$$

and $0 \neq 2$, so there is a contradiction. Thus we cannot find a T satisfying all of these conditions, and the system has no solution.