Section 10.3 Solutions

1. We would like to find $T(x,t)$, where $0 \le x \le \pi$, $t \ge 0$, and $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$. We have furthermore that $T(0,t) = T(\pi,t) = 0$, $T(x,0) = 4\sin x$, and

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \mu t} \sin nx$$
$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

To find the coefficients c_n, we need to find the Fourier series of $T(x,0)$; note that x ranges from 0 to π, which is not the ordinary domain for Fourier series expansion. So to do this, we first extend the function over the interval $[-\pi, 0]$ so that it is odd: $\Theta(x) = 4\sin x$, $-\pi \le x \le \pi$. The Fourier series for this function is easy to compute, since it is already expressed as a sum of sine functions: $\Theta(x) = \sum_{n=1}^{\infty} c_n \sin nx \Rightarrow c_1 = 4$, $c_n = 0$ for $n > 1$. So we have

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \mu t} \sin nx = 4e^{-\mu t} \sin x.$$

2. We would now like to solve for $T(x,t)$ where $0 \le x \le \pi$, $t \ge 0$, and $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$, with the boundary conditions $T(0,t) = T(\pi,t) = 0$ and

$$T(x,0) = \begin{cases} 0 & x \le \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < x < \frac{3\pi}{4} \\ 0 & x \ge \frac{3\pi}{4} \end{cases}.$$

Again, $T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \mu t} \sin nx$. To compute the c_n, we consider $T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$. Define $\Theta(x)$ to be equal to $T(x,0)$ over $[0,\pi]$ and an odd function over the entire interval $[-\pi, \pi]$:

$$\Theta(x) = \begin{cases} 1 & \frac{\pi}{4} < x < \frac{3\pi}{4} \\ -1 & -\frac{3\pi}{4} < x < -\frac{\pi}{4} \\ 0 & \text{otherwise on the interval } -\pi \le x \le \pi \end{cases}$$

We now find the Fourier series of $\Theta(x)$. In general, since $\Theta(x)$ is odd we may write it as $\Theta(x) = \sum_{n=1}^{\infty} c_n \sin nx$, where the coefficients c_n we determine as follows:

$$\begin{aligned} c_n &= \langle \Theta(x), \sin nx \rangle \\ &= \frac{2}{\pi} \int_0^{\pi} \Theta(x) \sin nx \, dx = \frac{2}{\pi} \int_{\pi/4}^{3\pi/4} \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_{\pi/4}^{3\pi/4} = \frac{2}{n\pi} \left(-\cos \frac{3\pi}{4} n + \cos \frac{\pi}{4} n \right) \end{aligned}$$

Notice that if n leaves a remainder of 1 or 7 after dovisopn by 8, $c_n = \frac{2\sqrt{2}}{n\pi}$; if n leaves a remainder of 3 or 5, $c_n = -\frac{2\sqrt{2}}{n\pi}$; and if n is even $c_n = 0$. So we have

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\mu t} \sin nx \ , \ c_n = \begin{cases} \frac{2\sqrt{2}}{n\pi}, & n = 1, 7 \bmod 8 \\ -\frac{2\sqrt{2}}{n\pi}, & n = 3, 5 \bmod 8 \\ 0, & n = 0 \bmod 2 \end{cases} .$$

3. If $T(x,t) = \frac{100}{\pi}x$, then $\frac{\partial T}{\partial t} = 0$, since $T(x,t)$ is constant in t, and $\frac{\partial^2 T}{\partial x^2} = 0$, since $T(x,t)$ is linear in x. Finally, the initial conditions are satisfied, as $T(0,t) = \frac{100}{\pi} \cdot 0 = 0$, and $T(\pi,t) = \frac{100}{\pi} \cdot \pi = 100$, so this is a solution of the differential equation.

4. Let $h(x)$ be a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial t^2}$, subject to the conditions $T(0,t) = 0$, $T(\pi,t) = 100$, and $T(x,0) = \frac{100}{\pi}x$. Let $s(x,t)$ be a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial t^2}$ subject to the conditions $T(0,t) = 0$, $T(\pi,t) = 100$, and $T(x,0) = -\frac{100}{\pi}x$. Then $h(x,t) + s(x,t)$ is a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial t^2}$ such that $T(0,t) = 0$, $T(\pi,t) = 100$, and $T(x,0) = 0$.

By Exercise 3 we can use $h(x,t) = \frac{100}{\pi}x$. To find $s(x,t)$ we can use the methods of this section. Let $\Theta(x) = -\frac{100}{\pi}x$; note that $\Theta(x)$ is an odd function over $[-\pi,\pi]$ and it equals $s(x,t)$ over $[0,\pi]$. From page 10 we see that

$$\Theta(x) = -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

so that

$$s(x,t) = \Theta(x) = -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2\mu t} \sin nx$$

and

$$s(x,t) + h(x,t) = \frac{100}{\pi}x - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2\mu t} \sin nx.$$

As $T \to \infty$, $e^{-n^2\mu t} \sin nx \to 0$, so that $T(x,t) \to \frac{100}{\pi}x$.

5. If $T(x,t) = e^{-n^2\mu t} \cos nx$, then $\frac{\partial T}{\partial t} = -n^2\mu e^{-n^2\mu t} \cos nx$ and $\frac{\partial^2 T}{\partial x^2} = -n^2 e^{-n^2\mu t} \cos nx$, so that $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$, and we see that

$$\frac{\partial T}{\partial x}(0,t) = -ne^{-n^2\mu t} \sin nx\big|_{x=0} = 0$$
$$\frac{\partial T}{\partial x}(\pi,t) = -ne^{-n^2\mu t} \sin nx\big|_{x=\pi} = 0$$

6. From Exercise 5 and by the lienarity of the differential equation and boundary conditions,we can assume that

$$T(x,t)=\sum_{n=0}^{\infty} c_n e^{-n^2\mu t}\cos nx = c_0+\sum_{n=1}^{\infty} c_n e^{-n^2\mu t}\cos nx$$

so that $T(x,0)=c_0+\sum_{n=0}^{\infty} c_n \cos nx = x$. We need to find a Fourier series for $T(x,0)$ with only cosine and constant terms. In this case define $\Theta(x)$ to be equal to $T(x,0)$ on $[0,\pi]$ and an *even* function on $[-\pi,\pi]$:

$$\Theta(x)=\begin{cases} x, & 0\le x\le \pi \\ -x, & -\pi \le x \le 0 \end{cases} \Rightarrow \Theta(x)=|x|.$$

From the assignment on Section 10.2, we know that

$$|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$$

so

$$T(x,t)=\sum_{n=0}^{\infty} c_n e^{-n^2\mu t}\cos nx = \frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{\infty} e^{-n^2\mu t}\frac{\cos(2k+1)x}{(2k+1)^2}$$

i.e., $c_0=\frac{\pi}{2}$, $c_n=-\frac{4}{\pi n^2}$ for n odd, and $c_n=0$ for n even and not equal to zero.