Section 10.2 Solutions

1. Since the set $\{1, \sin at, \cos bt\}$ is orthogonal, $|1+\sin t+3\cos 5t+2\sin 10t|| = \sqrt{2+1+9+4} = 4$.

2. We see that the two functions are orthogonal because their inner product is given by

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{2}}\frac{t}{\pi}\right)dt = \frac{3}{2\pi^2}\int_{-\pi}^{\pi} t\ dt = \frac{3}{2\pi^2}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0.$$

Since $||\frac{1}{\sqrt{2}}|| = 1$, we need only check $||\sqrt{\frac{3}{2}}\frac{t}{\pi}||$:

$$||\sqrt{\frac{3}{2}}\frac{t}{\pi}|| = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{3}{2\pi^2}t^2\ dt = \frac{1}{2\pi^3}\left[t^3\right]_{-\pi}^{\pi} = 1$$

So $\frac{1}{\sqrt{2}}$ and $\sqrt{\frac{3}{2}}\frac{t}{\pi}$ are indeed orthonormal, and form a basis for the subspace V.

The projection of t^2 onto V is then given by

$$\text{proj}_V(t^2) = \left\langle\frac{1}{\sqrt{2}}, t^2\right\rangle\frac{1}{\sqrt{2}} + \left\langle\sqrt{\frac{3}{2}}\frac{t}{\pi}, t^2\right\rangle\sqrt{\frac{3}{2}}\frac{t}{\pi} = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right)\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\frac{t}{\pi^2}\cdot 0 = \frac{\sqrt{2}\pi^2}{3}$$

3. We first find the Fourier coefficients for $\frac{1}{\sqrt{2}}$, $\cos nt$, and $\sin nt$:

$$\left\langle\frac{1}{\sqrt{2}}, |t|\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}|t|\ dt = \frac{\pi}{\sqrt{2}}$$

$$\langle\sin nt, |t|\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nt|t|\ dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nt\cdot t\ dt + \int_{0}^{\pi}\sin nt\cdot t\ dt = 0$$

$$\begin{aligned}\langle\cos nt, |t|\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nt|t|\ dt = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nt\cdot t\ dt + \int_{0}^{\pi}\cos nt\cdot t\ dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} t\cos nt\ dt = \frac{2}{\pi}\left[\frac{t}{n}\sin nt\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} sintnt\ dt\right] \\ &= \frac{2}{\pi}\left(\frac{1}{n^2}\right)(\cos n\pi - 1) = \begin{cases}0, & n = 2k; \\ -\frac{4}{\pi n^2}, & n = 2k+1\end{cases}.\end{aligned}$$

$$\Rightarrow |t| = \left(\frac{\pi}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \sum_{n\ \text{odd}}\left(-\frac{4}{\pi n^2}\right)\cos nt = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)t}{(2k+1)^2}$$

4. Integrating by parts, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{at}\cos nt\, dt &= \frac{-e^{at}\sin nt}{n}\Big|_{-\pi}^{\pi} + \frac{a}{n}\int_{-\pi}^{\pi} e^{at}\sin nt\, dt \\
&= \frac{a}{n}\left[\frac{e^{at}\cos nt}{n}\Big|_{-\pi}^{\pi} - \frac{a}{n}\int_{-\pi}^{\pi} e^{at}\cos nt\, dt\right] \\
&= \frac{a\cos nt e^{a\pi}}{n^2}\Big|_{-\pi}^{\pi} - \frac{a^2}{n^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{at}\cos nt\, dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_{-\pi}^{\pi} e^{at}\cos nt\, dt &= \frac{1}{n^2+a^2}\ ae^{at}\cos nt\Big|_{-\pi}^{\pi} = \frac{(-1)^n a\left(e^{a\pi}-e^{-a\pi}\right)}{n^2+a^2} \\
&= \frac{(-1)^n 2a}{n^2+a^2}\sinh\left(a\pi\right)
\end{aligned}$$

5. Since cosh is an even function (i.e., $\cosh(-at) = \cosh(at)$), we need only worry about the Fourier coefficients for the even terms of the series (the odd ones integrate to zero):

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2}}, \cosh at\right\rangle &= \frac{e^{a\pi}-e^{-a\pi}}{\sqrt{2}\pi a} \\
\langle \cos nt, \cosh at\rangle &= \frac{(-1)^n a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(n^2+a^2)} \qquad \text{from Exercise 4}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \cosh at &= \frac{e^{a\pi}-e^{-a\pi}}{\sqrt{2}\pi a} + \frac{a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt \\
&= \frac{a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi}\left[\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt\right]
\end{aligned}$$

6. Let $t = \pi$ in Exercise 5. Then we get

$$\begin{aligned}
\cosh a\pi &= \frac{e^{a\pi}-e^{-a\pi}}{\pi}\left[\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\right] \\
\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} &= \frac{\pi\cosh a\pi}{a\left(e^{a\pi}-e^{-a\pi}\right)} - \frac{1}{2a^2}
\end{aligned}$$