Section 10.1 Solutions

1. a. Let f and g be continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Although $f+g$ and cf are also continuous, so this is not a subspace of C^∞, as not all continuous functions are differentiable, and so not all elements of this set are contained in C^∞. Thus this set is not even a subset of C^∞, and hence it cannot be a subspace.

b. Let $S \subset C^\infty$ be the set of functions such that $f(0)+f'(0)=0$, and let $f,g \in S$. Then $(f+g)(0)+(f+g)'(0)=f(0)+g(0)+f'(0)+g'(0)=0 \Rightarrow (f+g)\in S$. Also, $cf(0)+(cf)'(0)=c(f(0)+f'(0))=0 \Rightarrow cf \in S$, so S is a subspace.

c. Let $A \subset C^\infty$ be the set of functions such that $f+f'=0$, and let $f,g\in A$. Then $(f+g)+(f+g)'=f+g+f'=g'=0 \Rightarrow (f+g)\in A$, and $(cf)+(cf)'=c(f+f')=0 \Rightarrow cf \in A$, so that A is also a subspace.

d. Let $B \subset C^\infty$ be the set of functions such that $f(0)=1$. If $f,g\in B$, then $(f+g)(0)=f(0)+g(0)=2 \Rightarrow (f+g)\notin B$, so B is not a subspace.

2. a. The set $\{1,t,t^2,t^3\}$ is linearly independent, for if $c_0+c_1t+c_2t^2+c_3t^3=0$ then we must have $c_0=c_1=c_2=c_3=0$.

b. Since $1(1+t+t^2)+(-1)t^2+1(1+t)=0$, the set is not linearly independent.

c. This set is linearly independent because there is no way to relate $\sin t$, e^t, and e^{-t}. (Note that if we had $\sinh t$ instead of $\sin t$, then this set would not be linearly independent, for $\sinh t = \frac{1}{2}(e^t-e^{-t})$).

d. Since $\sin(t+\frac{\pi}{3})=\sin t\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\cos t=\frac{1}{2}\sin t+\frac{\sqrt{3}}{2}\cos t$, this set is not linearly independent.

3. a. We see that $T(f+g)=(f+g)(0)=f(0)+g(0)=T(f)+T(g)$ and $T(cf)=(cf)(0)=c\cdot f(0)=cT(f)$, so T is linear.

b. Here $T(f+g)=(f+g)^2+(f+g)'=f^2+2fg+g^2+f'+g' \neq f^2 = f'+g^2+g'=T(f)+T(g)$, so T is not closed under addition, and is therefore not linear.

c. In this case $T(f+g)=\begin{bmatrix}(f+g)(0)\\(f+g)(1)\end{bmatrix}=\begin{bmatrix}f(0)+g(0)\\f(1)+g(1)\end{bmatrix}=\begin{bmatrix}f(0)\\f(1)\end{bmatrix}+\begin{bmatrix}g(0)\\g(1)\end{bmatrix}=T(f)+T(g)$, and $T(cf)=\begin{bmatrix}(cf)(0)\\(cf)(1)\end{bmatrix}=c\begin{bmatrix}f(0)\\f(1)\end{bmatrix}=cT(f)$, so T is linear.

d. Finally, $T(f+g)=\int_0^1(f+g)(t)dt=\int_0^1 f(t)dt+\int_0^1 g(t)dt=T(f)+T(g)$, and $T(cf)=\int_0^1 cf(t)dt=c\int_0^1 f(t)dt=cT(f)$, so T is linear.

4. We need to find the space of functions f such that $T(f)(t)=f''(t)-f(0)=0$. In order for this to be true, we must have $f''(t)=f(0)$. Now, the only functions

satisfying this expression are polynomials of the form $f(t) = \frac{a}{2}t^2 + bt + a$; a basis for this space is then $\{\frac{1}{2}t^2 + 1, t\}$.

5. Notice that $f(0)$ and $f'(0)$ are just arbitrary constants (just let $f(t) = at + b$). Therefore all elements of the image of T have the form $a + bt + (a+b)t^2$, so a basis for the image is $\{1 + t^2, t + t^2\}$.

6. Solving the equation $T(f) = \lambda f$, we see that $f' + f = \lambda f \Rightarrow f' = (\lambda - 1)f \Rightarrow f(t) = Ce^{(\lambda-1)t}$, so the eigenvalues λ may take on any value and the associated eigenspace is spanned by $e^{(\lambda-1)t}$.

7. The characteristic polynomial for the equation is $p_T(\lambda) = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3)$. The roots are distinct, so a basis for $\ker T$ is $\{e^{3t}, e^{-4t}\}$. To find such a function f, notice that $f \in \ker T \Rightarrow f(t) = Ae^{3t} + Be^{-4t}$. Setting $f(0) = 0$ implies $A = -B$; similarly $f'(0) = 0 \Rightarrow 3A = 4B \Rightarrow A = B = 0$, so $f(t) = 0$.

8. The characteristic polynomial is $p_T(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda - (-1-i))(\lambda - (-1+i))$. The roots of this equation are distinct, so a basis for the kernel of T is $\{e^{-(1+i)t}, e^{-(1-i)t}\}$. Again $f \in \ker T \Rightarrow f(t) = Ae^{-(1+i)t} + Be^{-(1-i)t}$, and $f(0) = 1 \Rightarrow A + B = 1 \Rightarrow A = 1 - B$, and $f'(0) = 1 \Rightarrow -(A+B) - i(A-B) = 1 \Rightarrow (1-B)(1+i) + B(1-i) = -1 \Rightarrow A = i + \frac{1}{2}, B = -i + \frac{1}{2}$, so that $f(t) = e^{-t}\left(\left(\frac{1}{2} - i\right)e^{-it} + \left(\frac{1}{2} + i\right)e^{it}\right) = e^{-t}(\cos t + 2\sin t)$.

9. a. Let the downward forces be positive, and let g be the acceleration due to gravity. Then

$$\begin{aligned}\text{weight of block} &= (\text{mass of block})g = (\text{density of block})(\text{volume of block})g \\ &= 800 \text{ g}\end{aligned}$$

and

$$\begin{aligned}\text{buoyancy} &= \text{weight of displaced water} = (\text{mass of displaced water})g \\ &= (\text{density})(\text{volume displaced})g = 1 \cdot 10^2 \cdot t \text{ g} = 100 \text{ g} \cdot t\end{aligned}$$

b. By Newton's Second Law of Motion, $m\frac{d^2x}{dt^2} = F =$ weight of block $-$ buoyancy $= 800g - 100g \cdot x(t)$. Here $m = 800$, so we have $\frac{d^2x}{dt^2} = g - \frac{g}{8}x(t) \Rightarrow \frac{d^2x}{dt^2} + \frac{g}{8}x = g$. The constant solution is $x_p = 8$, and the general solution is $x(t) = c_1 \cos\left(\sqrt{\frac{g}{8}}T\right) + c_2 \sin\left(\sqrt{\frac{g}{8}}t\right) + 8$. Now, the block is at rest at $t = 0$, so $c_2 = 0$, and $x(0) = 10 = c_1 + 8 \Rightarrow c_1 = 2$, so that the solution for this set of initial conditions is $x(t) = 2\cos\left(\sqrt{\frac{g}{8}}t\right) + 8$.

c. If a is the side length, the period is $\frac{2\pi}{\sqrt{g/pa}} = \frac{2\pi\sqrt{pa}}{\sqrt{g}}$, where p is the density. So the period increases if the density or size increases, or if the force due to gravity decreases. The initial consitions have no effect.

10. The characteristic polynomial is $p_T(\lambda) = \lambda^2 + 9$, so a basis for ker$T$ is e^{3it}, e^{-3it}. We need to then solve $T(f(t)) = \cos(\alpha t)$. There are two cases:

i. $\alpha \neq 3$. Then the particular solution is $f_p(t) = P\cos(\alpha t) + Q\sin(\alpha t)$. Substituting in to find P and Q, we get $f_p(t) = \frac{1}{9-\alpha^2}\cos(\alpha t)$, so that the general solution is $f(t) = c_1e^{3it} + c_2e^{-3it} + \frac{\cos(\alpha t)}{9-\alpha^2}$.

ii. $\alpha = 3$. What is a particular solution to $f''(t)+9f(t) = \cos(3t)$? Consider $f(t) = At\cos 3t + Bt\sin 3t$. Then $f''(t) + 9f(t) = -6A\sin 3t + 6B\cos 3t = \cos 3t \Rightarrow A = 0,\ B = \frac{1}{6}$. So $f_p(t) = \frac{t}{6}\sin 3t$, and the general solution is $f(t) = c_1\cos 3t + c_2\sin 3t + \frac{t}{6}\sin 3t$.

11. Let $t\frac{df(t)}{dt} = 1$. We see by substitution that $f_p(t) = \ln t$ is a solution. Suppose that $g(t)$ is another solution. Then

$$t\frac{d(f_p(t) - g(t))}{dt} = t\left[\frac{df_p(t)}{dt} - \frac{dg(t)}{dt}\right] = t\frac{df_p(t)}{dt} - t\frac{dg(t)}{dt} = 1 - 1 = 0.$$

Also, if $t\frac{d(f_p(t)-g(t))}{dt} = 0$ then $t\frac{dg(t)}{dt} = t\frac{df_p(t)}{dt} = 1$. So $g(t)$ is a solution if and only if $t\frac{d(f_p(t)-g(t))}{dt} = 0 \Rightarrow g(t) = f_p(t) + c$. Therefore the general solution is $g(t) = \ln t + c$, which is not defined for $t \leq 0$, so there is no solution to this differential equation in C^∞.

12. Here we have $T(f(t)) = tf'(t) + f(t)$, and $g(t) = tf(t)$. If $T(f(t)) = 0$ then $tf'(t)+f(t) = 0$. By the product rule, $g'(t) = f(t)+tf'(t)$, so $g'(t) = 0 \Rightarrow g(t) = c$, where c is some constant. But $g(t) = tf(t) = c \Rightarrow f(t) = 0 \Rightarrow \dim(\ker T) = 0$.