SECOND PRACTICE EXAM SECOND HOURLY **Math 21b, Fall 2003**

Name:

MWF10 Izzet Coskun
MWF11 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T **F** If A is a non-invertible $n \times n$ matrix, then $\det(A) \neq \det(\text{rref}(A))$.

Solution:
A as well as $\text{rref}(A)$ have determinant 0.

T F If the rows of a square matrix form an orthonormal basis, then the columns must also form an orthonormal basis.

Solution:
You have proven that $AA^T = 1_n$ is equivalent to $A^TA = 1_n$.

T F If the diagonal entries of an $n \times n$ matrix A are odd integers and all the entries not lying on the diagonal are even integers, then A is invertible.

Solution:
Compute the determinant. It has to be odd because the diagonal pattern produces an odd number and the other patterns produce even numbers. The determinant is an odd integer and an odd integer can not be zero.

T **F** A 2×2 rotation matrix $A \neq I_2$ does not have any real eigenvalues.

Solution:
The rotation by an angle π has the real eigenvalue -1 with algebraic multiplicity 2.

T F If A and B both have $\vec{v}$ as an eigenvector, then $\vec{v}$ is an eigenvector of AB.

Solution:
Check $AB\vec{v} = \mu\lambda\vec{v}$ if λ is the eigenvalue to $\vec{v}$ of A and μ is the eigenvalue of $\vec{c}$ with respect to B.

T **F** If A and B both have λ as an eigenvalue, then λ is an eigenvalue of AB.

Solution:
Already diagonal matrices can be used for counter examples, like $A = \text{diag}(2, 1/2), B = \text{diag}(1/2, 2)$. It is even not true for nonzero 1×1 matrices.

T **F** Similar matrices have the same eigenvectors.

Solution:
They have the same eigenvalues, not eigenvectors

T F — If a 3×3 matrix A has 3 independent eigenvectors, then A is similar to a diagonal matrix.

Solution:
According to one of the basic results

T F — If a square matrix A has non-trivial kernel, then 0 is an eigenvalue of A.

Solution:
$A\vec{v} = 0$ means also $A\vec{v} = 0\vec{v}$.

T F — If the rank of an $n \times n$ matrix A is less than n, then 0 is an eigenvalue of A.

Solution:
The kernel is then not trivial by the dimension formula.

T F — Two diagonalizable matrices whose eigenvalues are equal must be similar.

Solution:
Both matrices are similar to a common diagonal matrix.

T F — A square matrix A is diagonalizable if and only is A^2 is diagonalizable.

Solution:
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counter example. It is not diagonalizable but $A^2 = 0$ is.

T F — If a square matrix A is diagonalizable, then $(A^T)^2$ is diagonalizable.

Solution:
$S^1 AS = B$, then $S^{-1}A^2S = B^2$ and $S^T(A^T)^2(S^{-1})^T = (B^T)^2$ is diagonal.

T F — If a square matrix A has k distinct eigenvalues, then $\text{rank}(A) \geq (k-1)$.

Solution:
Because in that case, there are $k-1$ different eigenvalues which are not zero and so $(k-1)$ linearly independent eigenvectors implying that the image of A is at least $k-1$ dimensional.

T F — There exist matrices A with k distinct eigenvalues whose rank is strictly less than k.

Solution:
Just make sure that one eigenvalue is 0.

T F — If A is an $n \times n$ matrix which satisfies $A^k = 0$ for some positive integer k, then all the eigenvalues of A are 0.

Solution:
If not, we would have a nonzero eigenvalue λ with eigenvector $\vec{v}$ and $A^n\vec{v} = \lambda^n\vec{v}$ is nonzero so that A^n can not be zero.

T F — If a 3×3 matrix A satisfies $A^2 = I_3$ and A is diagonalizable, then A must be similar to the identity matrix.

Solution:
The matrix $A = -I_n$ is diagonalizable and satisfies $A^2 = I_3$, but A has different eigenvalues as the identity matrix and can not be similar to the identity matrix.

T F — A and A^T have the same eigenvectors.

Solution:
They have the same eigenvalues.

T F — If A and B are diagonalizable, AB is also diagonalizable.

Solution:
Take $A = \begin{bmatrix} 2 & 1/2 \\ 0 & 1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ are both diagonalizable but the product is the shear and not diagonalizable.

T F — The least squares solution of a system $A\vec{x} = \vec{b}$ is unique if and only if $\ker(A) = 0$.

Solution:
If $\ker(A) = 0$, you have seen the formula for the solution. If $\ker(A)$ is not trivial and $\vec{v}$ is in the kernel, then any least square solution $\vec{x}$ has also $\vec{x} + \vec{v}$ as a least square solution.

Problem 3) (10 points)

Find the volume of the three dimensional parallelepiped in four dimensions which is spanned by the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution:
Form the matrix A which has the three vectors as columns. The volume is $\sqrt{\det(A^T A)} = \sqrt{\det(\begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 3 & 3 \end{bmatrix})} = \sqrt{2}$.

Problem 4) (10 points)

Assume that A is a skew-symmetric matrix, that is, it is a $n \times n$ matrix which satisfies $A^T = -A$.

a) Find $\det(A)$ if n is odd.

b) What possible values can $\det(A)$ have if n is even?

c) Verify that if λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

Solution:
a) Because $\det(A^T) = \det(A) = (-1)^n \det(-A)$, the determinant is 0 if n is odd.
b) If n is even, then the determinant can be anything.
c) We know that λ is also an eigenvalue of A^T (because the characteristic polynomials of A and A^T are the same). If $A^T\vec{v} = \lambda\vec{v}$, then $A\vec{v} = -A^T\vec{v} = -\lambda\vec{v}$ so that $-\lambda$ is also an eigenvalue.

Problem 5) (10 points)

The recursion

$$u_{n+1} = u_n - u_{n-1} + u_{n-2}$$

is equivalent to the discrete dynamical system

$$\begin{bmatrix} u_{n+1} \\ u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix}.$$

a) Find the (real or complex) eigenvalues of A.

b) Is there a vector $\vec{v}$ such that $||A^n\vec{v}|| \to \infty$?

c) Can you find any positive integer k such that $A^k = I_3$?

Solution:
a) The eigenvalues are $1, -i, i$. Call the eigenvectors $\vec{a}, \vec{b}$ and $\vec{c}$.
b) No, there is no such vector. If $\vec{v} = a\vec{a} + b\vec{b} + c\vec{c}$ we can write down the explicit solution $A^n\vec{v} = a1^n\vec{a} + b(-i)^n\vec{b} + ci^n\vec{c}$.
c) Yes, take for example the eigenvector $\vec{c}$ to $\lambda = i$ and $k = 4$.

Problem 6) (10 points)

Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

a) Find $\det(A)$.

b) Find all eigenvalues whether real or complex of A and state their algebraic multiplicities.

c) For each real eigenvalue λ of A find the eigenspace and the geometric multiplicity.

Solution:
a) Q is orthogonal with determinant is -1, there is only one pattern and the signature of this pattern is (-1) because there are three inversions in the pattern.
b) The characteristic polynomial is $f_A(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$.
Each eigenvalue $1, -1, i, -i$ has algebraic multiplicity 1.
c) The vector $[1, 1, 1, 1]^T$ is an eigenvector to the eigenvalue 1, the vector $[1, -1, 1, -1]$ is an eigenvector to the eigenvalue -1. The geometric multiplicity is 1.

Problem 7) (10 points)

Find S and a diagonal matrix B such that $S^{-1}AS = B$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} .$$

Solution:
The eigenvector to the eigenvalue 1 is $[2, -2, 1]^T$.
The eigenvector to the eigenvalue 2 is $[0, -1, 2]^T$.
The eigenvector to the eigenvalue 3 is $[0, 0, 1]$.
The matrix S has the eigenvectors in the columns. The matrix B has the eigenvalues of A in the diagonal: so

$$S = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

Problem 8) (10 points)

Find the function of the form

$$f(t) = a\sin(t) + b\cos(t) + c$$

which best fits the data points $(0, 0), (\pi, 1), (\pi/2, 2), (-\pi, 3)$.

Solution:
To fit the function, set up the linear equations, for which we want to find the least square solution:

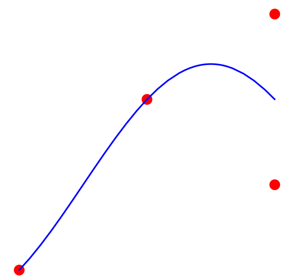
$$\begin{aligned} a\sin(0) + b\cos(0) + c &= 0 \\ a\sin(\pi) + b\cos(\pi) + c &= 1 \\ a\sin(\pi/2) + b\cos(\pi/2) + c &= 2 \\ a\sin(-\pi) + b\cos(-\pi) + c &= 3 \end{aligned}$$

which is

$$\begin{aligned} b + c &= 0 \\ -b + c &= 1 \\ a + c &= 2 \\ -b + c &= 3 \end{aligned}$$

Solution:
or in matrix form

$$A\vec{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b} .$$



We get the unknown coefficients

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} .$$

The solution is $1 - \cos t + \sin(t)$.

Problem 9) (10 points)

Let V be the image of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} .$$

a) Find the matrix P of the orthogonal projection onto V.

b) Find the matrix P' of the orthogonal projection on to $V^\perp$.

Solution:
a) Just use the formula $P = A(A^TA)^{-1}A^T$. We get

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

b) The formula is verified by applying P and P' to it. We have $P' = 1_4 - P$. So

$$\begin{bmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{bmatrix} .$$